The Limitless Mind of Abstract Thinking

How the mind leaves the here and now, who did it best, and where it can take us

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Why this article

I did not follow the usual path through higher education, and I was never trained to think in one specialised way. I believe that is why I was able to work in so many fields: technology, finance and mathematics, science, writing and the arts. Again and again I became skilled in a new area in months where it is expected to take years. I did it by looking for the structure underneath a subject and carrying it across from the last one.

That is my point, and it comes from experience. People have been fed knowledge a drop at a time. We now have intelligent systems that can open the whole of it at once, and that can assess and present what is known far faster than anyone could in the past. With them, anyone can learn at the rate I have described, provided the systems are used ethically and the thinking stays our own.

I have been making this argument for a long time. In 2000, as chief executive of Milinx, I spoke about the wireless future and what I called intelligent applications on the programme *World Business Review*, on which the internet pioneer Vinton Cerf was the expert panelist (Milinx, 2000).

> **“We are aware of what we are aware… not more.”** Maynard L. Dokken

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Introduction

A child of three is shown a small model of a room. A toy is hidden behind the model's tiny sofa. The child is then taken into the real room next door and asked to find the same toy in the same place. Most three-year-olds walk straight to the sofa. Most children six months younger cannot do it, although they can find the toy in the model itself without trouble (DeLoache, 2005).

Nothing about the younger child's eyes, memory or hands is lacking. What is missing is one step: seeing that the small room *stands for* the large one. Once that step is taken, a model, a map, a word, a number or an equation can all do the same thing. They let a mind work on something that is not in front of it.

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That step is abstract thinking. It is the subject of this article, and the claim made here is a large one: nearly everything our species has built that no other species has built, from geometry to the symphony to the computer, was built with it. The people we call the greatest minds in history were, almost without exception, people who took that step further than anyone before them.

The article has five parts.

- **Part I** is the science: what abstract thinking is, how it grows in a child, how clinicians measure it, what the brain does when it happens, what happens when it is lost, and whether it can be strengthened. It is told through the philosophers and scientists who worked it out, because the minds that could describe abstraction were themselves among its best examples.
- **Part II** is the record: twenty-five minds, from a grammarian in ancient India to the engineer who measured information, and the exact abstract move each one made.
- **Part III** is the range: the fields in which abstract thinking produces extraordinary work, with the arts and entertainment given their full place beside science.
- **Part IV** is about the newest tool of the abstract mind: what the evidence says about thinking with the help of artificial intelligence, when it helps and when it harms.
- **Part V** is a summary of the same thinking at work in a series of ventures over four decades, each of which will have an article of its own.

Many people, offered a new way to think, decline it before they have tried it. This article is written for those who would rather find out.

A word on the title. No mind is limitless in the literal sense, and Part I gives the measured limits. The word is used for something else that the evidence does support: abstraction is the one mental tool whose reach is not fixed by what the senses can take in. Each abstraction, once made, becomes a thing the next one can be built on. That is why the ceiling keeps moving.

One rule was kept throughout. Famous stories and famous quotations were checked against their sources, and a number of them did not survive. Where a story is legend, the article says so. Sorting the record from the legend is itself an exercise in the thinking this article is about.

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Part I. The abstract-thinking mind and how it works

1. What abstraction is

In brief: abstract thinking is thinking about what is not in front of you. It is made of at least six abilities. It is not made of words: much of it happens in images, in the sense that two things fit, and in what a life has taught.

1.1 A word without a single definition

The first surprise is that there is no agreed definition. The *Stanford Encyclopedia of Philosophy* states plainly that there is no standard account of where the line between the abstract and the concrete falls, even though everyone agrees on the clear cases: numbers are abstract, rocks and trees are concrete. The old view, it reports, treated abstraction as a mental process in which new ideas are formed by attending to what several things share and ignoring how they differ (Rosen, 2025).

Psychology is in the same position. A review of how the field uses the word found that definitions differ from one area of research to the next, while agreeing on what abstraction is good for: "broadening mental horizons, integrating new experiences, and communicating information to others" (Burgoon et al., 2013). A more recent theory proposes that the mind holds representations at several levels, from those tied to one sense, through those that combine the senses, up to categories that depend on language and social life, and that the whole arrangement exists so that we can predict what is not yet in front of us (Gilead et al., 2020).

For this article a working description is enough. **Abstract thinking is thinking about what is not present to the senses: a property apart from the thing that has it, a relation apart from the things related, a category apart from its members, a possibility apart from what actually happened.** The definition used in this article rests on six abilities, each studied in its own right by the researchers named below. The list is a summary of what has been studied so far. It is not a limit, and there are surely others. After the six comes a point that is easy to miss, and that changes how the whole subject looks: none of them depends on language.

1.2 Six abilities, at least

Seeing relations. Two situations are analogous, in the definition used by the psychologist Keith Holyoak, "if they share a common pattern of relationships among their constituent elements", even when the elements themselves have nothing in common (Holyoak, 2012). Hand is to finger as foot is to toe. The inference rests on the role a thing plays and not on what it looks like. Holyoak calls analogy the prime example of relational reasoning, and it will turn out to be the single most common move in the discoveries of Part II.

Generalising. In 1987 Roger Shepard proposed a law for the most basic step of all: carrying what was learned about one case over to another. The chance that a learned response transfers to a new case falls off in a regular way as the new case becomes less similar to the old one. Shepard derived the law from an argument about how any organism should reason in a world of natural kinds (Shepard, 1987).

Using symbols. This is the three-year-old and the model room. Judy DeLoache, who ran those studies, explains the younger children's failure by what she calls dual representation: a symbol is both a thing in its own right and a stand-in for something else, and a very young child cannot hold both at once. In one experiment she told children of two and a half that a machine had *shrunk* the room. Now the small room was simply the big room made small, with no standing-for involved, and the children found the toy (DeLoache, 2005).

Forming concepts with no single thing to point at. "Cat" has an edge and a shape. "Freedom" does not. A review of the research found that abstract concepts are less stable over time, vary more from person to person, and are shaped more by culture than concrete ones. It concluded that they are learned through both bodily experience and language, and that social skills matter more for them, because nobody can point at justice (Borghi et al., 2017).

Thinking about what did not happen. People readily imagine how things could have gone otherwise. Ruth Byrne's research shows this imagining follows rules. People mentally undo actions more easily than failures to act, events they controlled more easily than events they did not, and the most recent event in a chain more easily than earlier ones. Her conclusion is that imagination and reason are not opposites: both work by considering possibilities (Byrne, 2007).

Thinking about one's own thinking. John Flavell opened this field in 1979 under the name metacognition. He described the knowledge a person has about what affects their own thinking, and the feelings that go with thinking, such as sudden puzzlement or the sense of being close to understanding (Flavell, 1979). It is abstraction turned on itself, and without it no one could deliberately improve the way they think.

1.3 Abstract thinking is not made of words

Most accounts of abstract thinking, including much of the research described in this article, are about words, numbers and symbols. That is partly because words are what can be tested and written down. It gives a false picture. For many of the people who have used this kind of thinking most powerfully, the words came last.

> **"From life experiences we can create so much."** Maynard L. Dokken

Thinking in images. Albert Einstein was asked by the mathematician Jacques Hadamard how he thought. He answered that words "do not seem to play any role in my mechanism of thought". The elements of his thinking were images that he could call up and combine at will, and the search for words came afterwards, with effort (Hadamard, 1954). Michael Faraday, who knew almost no mathematics, pictured the space around a magnet as filled with lines of force, and physics was rebuilt on what he saw (Part II). The historian of engineering Eugene Ferguson argued that an engineer's design takes shape first in the mind's eye, by a kind of thinking that is visual and cannot be reduced to words or calculation (Ferguson, 1977).

Thinking in wholes. Susanne Langer drew the distinction that makes room for this. Words and proofs are taken in one piece at a time. A picture, a piece of music or a situation is grasped all at once, and it can carry a structure that no sentence could hold (Windle, n.d.). The sudden insight studied in the laboratory arrives in the same way: complete, and before the person can say how they reached it (Kounios & Beeman, 2014).

> **"Imagination is more important than knowledge."** Albert Einstein, 1929 (Quote Investigator, 2013)

Thinking from experience. Vygotsky observed that the concepts a person builds from their own life grow upward from experience, and are strong exactly where the concepts taught in school are weak: they are full of content (Vygotsky, 1934/1987). The chess master who sees a formation where a beginner sees pieces has not been told anything the beginner was not told. Years of experience have changed what the master perceives (Chase & Simon, 1973). One theory holds that all concepts work this way, as partial re-runs of what was seen, done and felt (Barsalou, 1999).

Thinking with the hands. Architects at work do not draw ideas they already have. They draw, see something in the drawing they did not expect, and respond to it (Goldschmidt, 1991). The thought is in the doing.

These four are examples of what is possible and not a complete list. What they have in common is that the structure is grasped first and described afterwards, if it is described at all. A person can see that two situations have the same shape, that a plan will not work, or that something is missing from a design, without being able to say why for some time. That is abstract thinking. It draws on everything a person has lived through, and it is open to people who have never thought of themselves as abstract thinkers, because they have never been good with the words.

Next: the question of how a mind gets from particular things to general ideas was the central question of philosophy for 2,400 years. Twelve thinkers took positions on it, and modern science has ended up testing nearly all of them.

2. The pioneers and philosophers who mapped it

In brief: Plato held that the abstract is the most real thing there is. Aristotle held that it is drawn out of the concrete by leaving things out. The argument that followed, over whether general ideas exist at all, has never ended, and it is now being settled in the laboratory.

Long before there were laboratories, the question of how a mind gets from particular things to general ideas was the central question of philosophy. The twelve thinkers below did not agree with one another. Read in order, their disagreement is a 2,400-year investigation, and modern science has ended up testing most of the positions they staked out.

2.1 Plato: the abstract is the most real

Plato's answer was the boldest. In the *Phaedo*, the *Republic* and the *Symposium* he describes Forms: immaterial, unchanging and timeless. A beautiful thing is beautiful because it shares in Beauty itself, and the things we see are imperfect copies. His image for this, in the *Republic*, is the cave, where prisoners take shadows on a wall for reality until one of them is led up into the sunlight. Learning, for Plato, is the soul recollecting Forms it knew before birth (Silverman, 2014).

Few people now hold Plato's view as he stated it. Its importance is that he was the first to say that the objects of abstract thought are not weaker than the things we touch. For a mathematician proving something about every triangle that could ever exist, that still feels true.

2.2 Aristotle: the abstract is drawn out of the concrete

Plato's student turned the picture over. For Aristotle, mathematical objects are reached by *removing*: one starts with a physical body and takes away, in thought, everything irrelevant to the question, such as its matter and the fact that it can be perceived. His Greek word for this removal, *aphairesis*, became the Latin *abstractio*, which is where our word comes from. A geometer studies a bronze triangle only *as* a triangle. Aristotle added that the more a science removes, the more exact it is: arithmetic is more exact than geometry, and geometry more exact than the study of motion (Mendell, 2004).

This is the position most scientists have worked from ever since: abstraction as deliberate neglect.

2.3 Locke, Berkeley and Hume: is there such a thing as a general idea?

John Locke, in 1689, described the mind forming general ideas by dropping the particular time and place that tie an idea to one individual. Such abstract ideas, he held, are what our general words stand for (Uzgalis, 2022).

George Berkeley thought this impossible and said so with relish. Locke had written that the general idea of a triangle must be, in the words Berkeley quotes, "neither oblique nor rectangle, neither equilateral, equicrural, nor scalenon, but all and none of these at once". No one, Berkeley replied, can form such an idea. His alternative was that an idea which is itself particular "becomes general by being made to represent or stand for all other particular ideas of the same sort" (Berkeley, 1710).

David Hume called Berkeley's view one of the great discoveries of his time and added a mechanism: habit. A particular idea is attached to a general word, and the word, by custom, calls up other instances when they are needed. "Abstract ideas are therefore in themselves individual, however they may become general in their representation" (Hume, 1739/2007).

This argument has never ended. It is, almost word for word, the modern dispute in cognitive science over whether the mind stores abstract summaries or particular examples, and whether concepts are built from sensory experience or are something apart from it (section 6.2).

2.4 Kant: the mind supplies the framework

Immanuel Kant's *Critique of Pure Reason* (1781) made a different kind of claim. Experience, he argued, needs two things: what the senses deliver, and concepts to organise it. In the sentence the *Stanford Encyclopedia* gives as "Concepts without intuitions are empty, intuitions without concepts are blind," each is useless without the other (Wuerth & Brook, 2020). Some concepts are not learned from experience at all, because they are what makes experience possible. He listed twelve of these categories, among them unity, plurality, causality and possibility.

Kant also saw the practical problem that follows. How does a wholly abstract concept such as cause ever get applied to what we see? His answer was the *schema*: a rule, produced by the imagination, that translates an abstract concept into a pattern in time. The schema of cause is one event regularly following another (Hanna, 2017). Psychology later borrowed the word.

2.5 Peirce: how a new idea enters the world

The American philosopher Charles Sanders Peirce asked a question the others had passed over: where do *new* ideas come from? Deduction only unpacks what is already assumed, and induction only generalises what has been seen. He named a third kind of inference, abduction, and gave its form: "The surprising fact, C, is observed. But if A were true, C would be a matter of course." So there is reason to suspect A. Abduction, he wrote, "is the only logical operation which introduces any new idea" (Douven, 2025). Every hypothesis in Part II began this way.

Peirce also named the operation that turns a description into an object. "Honey is sweet" becomes "honey possesses sweetness", and now sweetness is a thing one can reason about, compare and measure. He called this hypostatic abstraction and credited it with half the power of mathematics (Bergman & Paavola, n.d.). Velocity, energy, number and information all entered thought this way.

2.6 Frege: abstraction made exact

Gottlob Frege gave the idea its modern logical form. In 1879 he published the first system of modern formal logic. In 1884 he argued that a statement of number is a statement about a concept: to say there

are eight planets is to say something about the concept *planet*. He then defined sameness of number without mentioning numbers at all: two collections have the same number when their members can be paired off one to one (Zalta, 2026). The method, defining a new abstract object through a relation among familiar things, is still called an abstraction principle. Georg Cantor would use the same pairing-off to compare infinities (Part II).

In 1902 a letter from Bertrand Russell showed that one of Frege's basic laws led to contradiction (Zalta, 2026). It is a fair warning that abstraction carried out with full rigour can still go wrong.

2.7 Whitehead: the warning

Alfred North Whitehead, Russell's co-author on *Principia Mathematica*, supplied the caution every enthusiast for abstraction needs. In 1925 he named the "fallacy of misplaced concreteness": mistaking an abstraction for the concrete reality it was drawn from. The task of philosophy, he held, is to be the critic of abstractions, giving each its proper place and checking it against fuller experience (Irvine, 2022). A model is not the thing modelled. The best abstract thinkers in Part II knew exactly what their idealisations left out.

2.8 Cassirer and Langer: the symbolic animal

Ernst Cassirer argued in 1910 that the traditional picture, in which concepts are formed by climbing from particulars to what they share, misses how modern science actually works: its concepts are relations, functions and orderings. In *The Philosophy of Symbolic Forms* (1923–1929) he widened the argument. Myth, language and science are each a system of symbols that human beings place between themselves and the world (Friedman, 2022).

Susanne Langer carried this into the arts. In *Philosophy in a New Key* (1942) she held that language is only one kind of symbolism. Words and proofs are *discursive*: they are read in sequence and their parts have meaning separately. A painting or a piece of music is *presentational*: it is grasped as a whole. In a 1951 essay she drew the contrast that Part III depends on. Scientific abstraction generalises. Artistic abstraction produces a single object that embodies a form of feeling (Windle, n.d.). Both are abstraction.

2.9 Wittgenstein: no single thing in common

Ludwig Wittgenstein delivered the last major correction. Consider everything we call a game, he wrote. There is no one feature they all share. What we find instead is "a complicated network of similarities overlapping and criss-crossing", which he called family resemblance (Matar, 2026). Concepts do not need sharp edges to work. Twenty years later this remark became an experimental programme in psychology (section 6.2).

Taken together, the philosophers left a list of questions: whether the abstract is found or made, whether general ideas exist or only particular ones put to general use, what the mind brings to experience, where new ideas come from, and what abstraction costs. The rest of Part I shows what happened when these questions were put to children, patients and brain scanners.

Next: a baby of seven months, who cannot speak, can already tell "the same" from "different". And in the last hundred years whole nations have learned to think more abstractly than their grandparents did.

3. How it grows: in a child, in civilisation, and the universe

In brief: abstraction is present before speech. It is drawn out by teaching and by the habits of a culture, and whole populations have gained it within living memory. The section ends with a question about how far it reaches.

3.1 Before words

Abstraction begins earlier than anyone expected. In a 2015 study, babies of seven and nine months were shown several pairs of toys. Some babies saw pairs that matched; others saw pairs that differed. Then they were shown new pairs made of toys they had never seen. Of the 64 infants, 49 (77%) looked longer when the *relation* changed, from same to different or the reverse. They had picked up “same” and “different” as such, months before their first word (Ferry et al., 2015).

Two details matter. One example pair was not enough: the babies needed several, which suggests they learn the relation by comparing. And when individual toys were made too interesting beforehand, the learning failed. The object got in the way of the relation. A later study found the same ability, under narrower conditions, at three months (Anderson et al., 2018).

3.2 Piaget: the adolescent discovers the possible

The classic account of how abstract thought develops is Jean Piaget’s. With Bärbel Inhelder he interviewed more than 1,500 children and adolescents as they worked on physics problems, and proposed that at around 11 or 12 a new kind of thinking begins. He called it formal operations. Its mark is reasoning from possibilities: the adolescent can start from a hypothesis, including one that is false or has never been observed, and work out what would follow (Amsel & Moshman, 2015.

Later research kept the phenomenon and loosened the timetable. Reviewers conclude that these developments are “neither universal nor inevitable”. They depend on schooling and on whether a culture has use for them, and even young children reason hypothetically when the subject is familiar or the setting is pretend play (Amsel & Moshman, 2015. In one set of figures, only about 20% of British fourteen-year-olds in the mid-1970s performed at Piaget’s formal level (Flynn, n.d.). Abstract reasoning of this kind is an achievement, and not everyone is led to it.

3.3 Innate, and taught

> **“It is either innate or taught.”** Maynard L. Dokken

The infants of section 3.1 settle one half of an old argument. The capacity is there at the start, before any teaching. What happens to it afterwards depends on what a child meets. Lev Vygotsky, working in the Soviet Union in the 1920s and 1930s, supplied what Piaget’s account lacked: other people. He distinguished two kinds of concept. Everyday concepts grow upward out of experience. A child knows what a brother is long before being able to define one. Scientific concepts travel the other way. They begin as a definition inside an organised system, taught in school, and work downward toward concrete cases. Each kind is weak where the other is strong, and Vygotsky warned that a scientific concept taught without roots in the child’s own experience produces only empty words (Vygotsky, 1934/1987).

His best-known idea is the zone of proximal development: the distance between what a child can do alone and what the same child can do with guidance from an adult or a more capable companion (Vygotsky, 1978). On this view the innate capacity is extended from mind to mind before the extension is owned by one. Part IV returns to the idea, because a thinking tool can stand in the place of that more capable companion.

Jerome Bruner gave the order in which understanding is usually built: first through action, then through images, then through symbols (Bruner, 1966). Annette Karmiloff-Smith added the process that drives it. Knowledge that at first exists only inside a skill, as something a child can do, is represented again and again in the mind, each time in a more explicit form, until the child can inspect it, talk about it and carry it to a new problem (Karmiloff-Smith, 1994).

3.4 A whole century learns to think abstractly

The most striking evidence that abstraction can be acquired comes from entire populations. In 1987 James Flynn showed that scores on intelligence tests had risen by between 5 and 25 points in a single generation in fourteen countries (Flynn, 1987). A later analysis of 271 samples and almost four million people in 31 countries confirmed the rise across a century and measured where it was largest: 0.41 points a year on tests of fluid reasoning, against 0.21 on tests of acquired knowledge (Pietschnig & Voracek, 2015).

Flynn's explanation is the important part. The gains were concentrated in the most abstract tests. Between 1947 and 2002, Americans gained about 24 points on the Similarities test, which asks how two things are alike, and only about 4 points on vocabulary. Asked what a dog and a rabbit have in common, a person of 1900 would be likely to answer that you use a dog to hunt rabbits. The answer that scores highest today is that both are mammals. Flynn's phrase for the change was that modern people had put on "scientific spectacles": the habits of sorting the world into categories and of taking hypothetical questions seriously, spread by schooling and by the kind of work people now do. He insisted that our ancestors were not less intelligent. Their intelligence was applied to the concrete problems in front of them (Flynn, n.d.).

The causes of the rise are still debated. Its lesson for this article is not. The habit of abstraction is not fixed at birth. It spread through whole societies within living memory, and nothing suggests that the spreading is finished.

> **"We have the opportunity to make the next leap forward."** Maynard L. Dokken

3.5 And the universe

The section so far has followed abstraction from an infant to a century. The author's own thinking has for many years gone one step further, and it is set down here as a hypothesis and not as a finding.

The ability described in this article is, at bottom, the ability to grasp connection: that one thing stands for another, that two situations share a structure, that many cases fall under one law. The question is whether connection is only something minds impose, or whether minds are able to grasp it because it is there. The hypothesis is that there is a fabric of connection in the universe that holds all of it together, beyond what is known today, and that the abstracting mind is one of the places where it shows. The physical side of that idea, with the tests so far carried out on it, is set out in a separate paper on this site, *The Fabric Continuum Theory*.

> **"We do not know what we do not know."** Maynard L. Dokken

Science and knowledge are young. Every one of the twenty-five minds in Part II overturned something that the best-informed people of the time held to be settled. It would be strange if ours were the first generation with nothing of that kind still ahead of it.

Next: how a clinician decides whether a mind can still think abstractly, what those measurements can and cannot tell, and an account from someone the measurements were wrong about.

4. How clinicians measure it

In brief: the clinical study of abstract thinking began with soldiers wounded in the head. The tests built since then each capture one face of it. They measure something real, and they measure it imperfectly.

4.1 Goldstein's soldiers

The clinical study of abstract thinking began with the wounded of the First World War. The neurologist Kurt Goldstein ran an institute in Frankfurt for soldiers with brain injuries. Working with the psychologist Adhémar Gelb, he noticed that what many of them had lost was not memory or speech but a stance toward the world. Goldstein called it the abstract attitude: the ability to step back from a situation and consider it, instead of simply reacting to it. Patients without it were held to the concrete and the immediate (Frisch & Métraux, 2021).

Goldstein also described what he called the catastrophic reaction, the acute distress of a patient faced with a task once easily managed (Stahnisch, 2024). In 1941, by then a refugee in the United States, he and Martin Scheerer published a set of tests for abstract and concrete behaviour, including sorting tasks that are the ancestors of tests still in use (Goldstein & Scheerer, 1941).

His most famous case, a young soldier known as Schneider, has been disputed ever since: later critics suspected that the symptoms were exaggerated. A modern re-examination concluded that the original report was not watertight but that the patient probably did have a real disorder (Marotta & Behrmann, 2004). Goldstein's term has been replaced by newer ones. His observation, that damage to the brain can leave a person able to act but unable to step back, has not.

4.2 The tests

Each of the standard tests isolates one face of abstraction.

Raven's Progressive Matrices shows a grid of patterned figures with one missing. The person must work out the rules that run along the rows and columns. A detailed study of how people solve it found that what separates the best performers is the ability to induce more abstract relations and to keep track of more goals at once in working memory (Carpenter et al., 1990).

Similarities, part of the Wechsler intelligence scales, asks how two things are alike. It is scored so that naming a shared category earns more than naming a shared use (Flynn, n.d.). It rewards abstraction directly.

Proverb interpretation is used at the bedside in psychiatry. To explain why people in glass houses should not throw stones, a person must leave the houses and the stones behind. Concrete answers are common in several conditions, and for that reason the test cannot tell one condition from another (Kiang et al., 2007).

The Wisconsin Card Sorting Test, introduced in 1948, requires the person to discover a sorting rule from feedback alone, and then to notice, without being told, that the rule has changed. The tell-tale error is to go on sorting by the old rule. Patients with damage to the frontal lobes make that error far more often than patients with damage further back (Demakis, 2003).

The psychologist Raymond Cattell gave these abilities their place in the larger picture. He divided intelligence into two kinds. **Fluid intelligence** is reasoning on problems never met before. **Crystallised intelligence** is the knowledge built up through education and experience (Cattell, 1963). Raven's test is the classic measure of the first.

4.3 What the tests cannot do

None of these tests is pure. Every one of them also draws on working memory. Scores shift with schooling and culture, as the Flynn effect shows, so a test said to be free of culture is not. The same low score can have quite different causes. And a test that is sensitive to frontal damage is not thereby a test *of* the frontal lobes: a review of the evidence found no simple one-to-one link (Alvarez & Emory, 2006). The tests measure a real thing, imperfectly.

4.4 When the measurements are wrong: the author's account

[This passage is the author's own account. It is one person's experience and not a clinical finding.]

I can speak about section 4.3 from the inside.

In October 2009 I was struck by a drunk driver in Toronto and suffered injuries that were classed as catastrophic. I was assessed by medical and legal professionals of the highest standing. Their determination, which is on the record, was that I would never function as I had before. I was being prepared for a life in an institution.

What changed the course of it was not a treatment. I was on a streetcar, and I watched a man with a severe mental disability get on with his attendant. I said to myself: **I was never going to be that person.**

I told very few people about that moment, because I expected it to be used as evidence against me and not as a clue to how recovery happens. I set about rebuilding my thinking, piece by piece, and I asked for an independent evaluation by a specialist in neuroscience. It took a long time to have the restrictions that had been placed on my life removed. They were removed.

What I did has a name in the research, and it is the reason the account is in this article. The brain is not fixed after an injury. It reorganises itself in response to what it is made to do, and this capacity, neuroplasticity, lasts through life. A much-cited summary of the research sets out the conditions under which it happens. Function that is not used is lost. Function that is used, with effort, improves. Repetition matters, and intensity matters. And the work has to *matter to the person doing it*: tasks that are meaningful and engaging produce more change than tasks that are not (Kleim & Jones, 2008.

That last condition is where belief and will come in. Nothing in the research says that belief repairs tissue. What it shows is that belief decides whether the work gets done. In studies of recovery after stroke, patients with more confidence in their own ability, and more motivation at the start, were more independent at the end, and the main reason appears to be that they engaged more, persisted longer and did more of the practice that drives the change (Gangwani et al., 2022). A person who has been told there is no point, and who accepts it, stops supplying the brain with what it needs to rebuild. A person who refuses to accept it keeps supplying it.

I refused. I have rebuilt a mind, and I know that it can be done. I also know that a prediction made about a person, by people with authority, can become the thing that prevents their recovery if they believe it.

Next: what the brain is actually doing when a person thinks abstractly, including cells that respond to an idea and a map-making system that works on thoughts as it works on places.

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5. The brain that does it

In brief: one system at the front of the brain joins relations together. Another, at the sides, holds concepts. A third, deep inside, maps ideas as it maps places. New ideas arrive when these systems work together, often when attention is elsewhere.

For most of the twentieth century, abstract thinking could be studied only from the outside: through what children said and what patients could no longer do. From the 1990s it became possible to watch the healthy brain at work. The picture that has formed has four main parts.

5.1 The front of the brain joins relations together

In 1999 a team including James Waltz, Barbara Knowlton and Keith Holyoak compared two groups of patients: one with damage to the prefrontal cortex, behind the forehead, and one with damage to the front of the temporal lobes, beside the temples. The groups were matched for overall intelligence. On reasoning problems that needed only one relation to be held in mind, both groups managed. When two relations had to be combined, the prefrontal patients failed badly while the temporal patients did not. The prefrontal patients were nonetheless *better* than the others on tests of memory and of knowledge (Waltz et al., 1999).

The finding separated two things that had been run together. Knowing things is one capacity. Joining relations into a new conclusion is another, and it depends on the front of the brain.

Brain imaging then located it more exactly. Kalina Christoff and colleagues gave healthy adults matrix puzzles of the Raven's type that required tracking no relation, one, or two. Activity at the very front of the prefrontal cortex rose only at the step from one relation to two, the point where relations must be combined (Christoff et al., 2001). Silvia Bunge's group found the same region at work in verbal analogies: one part of the frontal lobe retrieved the meanings of the words, and the frontmost part did the comparing of one relation with another (Bunge et al., 2005).

5.2 A gradient from the concrete to the abstract

The frontal lobes also appear to be laid out along a line. Étienne Koechlin and colleagues proposed in 2003 that control of behaviour runs from the back of the frontal lobe to the front: at the back, action is governed by what is directly in front of us; further forward, by the present context; furthest forward, by what happened earlier and what is planned (Koechlin et al., 2003).

David Badre and Mark D'Esposito tested the idea with four experiments in which the rule governing a response was made more abstract step by step. Activity moved forward, toward the frontal pole, in step with the level of abstraction (Badre & D'Esposito, 2007). In their review two years later they added that influence seems to run mainly from front to back, and that damage further forward impairs abstract control while leaving concrete control intact. They were careful to say that whether this is a strict hierarchy is still being tested (Badre & D'Esposito, 2009).

5.3 A hub where concepts are held

Where is the meaning of a word such as *dog* kept, given that seeing one, hearing one and reading the word all lead to the same idea? The leading account is called hub-and-spoke. Knowledge tied to each sense is spread across the brain, and a hub at the front of both temporal lobes binds it into a single concept that can be generalised to new cases (Patterson et al., 2007; Lambon Ralph et al., 2017).

The strongest evidence is a disease. In semantic dementia the front of the temporal lobes wastes away, and conceptual knowledge erodes with it: what a word means and what an object is are lost across every sense together (Hodges & Patterson, 2007. Briefly disrupting the same region in healthy volunteers with magnetic stimulation produces a mild, temporary version of the same difficulty (Lambon Ralph et al., 2017).

Set beside the prefrontal patients of section 5.1, this gives a clean division of labour. One system holds the abstractions. Another combines them.

5.4 Cells that respond to an idea

In 2005 Rodrigo Quian Quiroga and colleagues reported recordings from single neurons in eight patients who had electrodes implanted for the treatment of epilepsy. They found cells deep in the temporal lobe that responded to one person or one place and to almost nothing else. One neuron fired to very different

photographs of the same actress. Another fired to pictures of the Sydney Opera House. In some cases the cell also fired to the written name (Quian Quiroga et al., 2005).

A cell that answers to a face, a drawing and a printed word alike is not answering to an image. It is answering to an idea. The caution is that each neuron was tested on a limited set of pictures, and no one claims there is one cell for each concept.

5.5 The brain's map-making, turned on ideas

A network deep in the brain, centred on the hippocampus, is known for building maps of physical space. Some of its cells fire in a regular grid as an animal moves around. In 2016 a group in Oxford asked whether the same system maps *ideas*. Volunteers learned about cartoon birds that varied in two ways, the length of the neck and the length of the legs. While they imagined one bird changing into another, their brain activity showed the six-fold pattern that grid cells produce, in the same regions used for finding one's way. People in whom the pattern was stronger did better at the task (Constantinescu et al., 2016).

The authors noted that a scanner cannot confirm what single cells are doing. The proposal that has grown from this work is nonetheless striking: that the machinery evolved for finding one's way through space is a general mechanism for organising knowledge of any kind, and for inferring what has never been directly experienced (Behrens et al., 2018). It would explain why we speak of ideas that are close together or far apart.

5.6 Number

Number is the purest everyday abstraction: three sheep, three notes and three wishes share nothing except threeness. Stanislas Dehaene and colleagues reviewed the evidence from imaging and from patients and identified a strip of the parietal lobe that is active whenever quantities are handled, whether they arrive as digits, as words or as groups of dots (Dehaene et al., 2003). The brain's sense of quantity does not depend on notation. The symbols came later and were attached to it.

5.7 Insight, creativity and sleep

Three further findings bear on how new abstractions arrive.

Insight. The sudden "aha" has a signature in the brain. Reviewing a decade of studies, John Kounios and Mark Beeman define insight as a sudden reinterpretation that brings out a meaning which was not the obvious one. It is preceded by attention turning inward, away from the outside world (Kounios & Beeman, 2014).

Creativity. Roger Beaty and colleagues scanned 163 people while they thought of unusual uses for ordinary objects. The originality of the ideas could be predicted from how strongly three brain networks were connected: the default network, active when the mind wanders; the executive network, which directs and evaluates; and a third that switches between them. These networks usually work in opposition. In highly creative people they work together. The laboratory scores were related to real creative achievement in the arts and sciences (Beaty et al., 2018).

Sleep. In a German experiment, volunteers practised a number task that contained a hidden shortcut. After eight hours of sleep, 13 of 22 found the shortcut. Of those who had stayed awake for the same eight hours, by night or by day, 5 of 22 did. Sleep helped only those who had worked on the problem first (Wagner et al., 2004). The mind goes on abstracting when attention is elsewhere, but only on material it has already been given.

Two popular beliefs should be set aside here. The notion that we use only a tenth of our brains has no basis. And the idea that creative people are "right-brained" does not survive the evidence above: creative thought draws on networks across both sides.

5.8 Are we the only ones?

Other animals abstract more than was once believed. Honeybees can learn the rule of choosing whichever item matches a sample, and apply it to things they have never encountered, even in a different sense (Giurfa et al., 2001). Crows trained only to match identical items went on, without further training, to match *relations*, choosing a pair that was related in the same way as a sample pair (Smirnova et al., 2015). Ducklings a day old, having followed a pair of identical objects, preferred new pairs that were also identical (Martinho & Kacelnik, 2016).

Yet a careful review argues that a real gap remains. Animals are excellent at relations they can perceive. There is little evidence that any of them reason about relations between relations, or about the roles things play in a system, which is what human analogy does without effort (Penn et al., 2008). The question is still open. If a line is to be drawn anywhere, it is there: the difference is less in abstracting than in building abstractions on top of abstractions.

Next: the mental moves themselves. Why people who hold the key to a problem usually fail to use it, and what experts see that beginners do not.

6. How it works: the mechanisms

In brief: analogy carries a structure from one thing to another and leaves the surface behind. Categories have a centre and not an edge. Expertise is a change in what a person sees.

6.1 Analogy maps relations and drops appearances

The most developed theory of how analogy works is Dedre Gentner's, published in 1983. To draw an analogy, the mind carries the *relations* over from a familiar case to an unfamiliar one and leaves the features of the objects behind. When the atom was compared to the solar system, what carried over was that a small body moves around a more massive one that attracts it. That the Sun is hot and yellow did not carry over, and nobody supposed it should. Gentner added a principle of selection: we prefer to carry over relations that hang together as a system, especially those tied by cause and effect (Gentner, 1983).

A famous experiment shows both the power of analogy and its weakness. People read a story about a general who captures a fortress by sending small groups of soldiers along many roads at once so that they converge on it. They were then given a medical problem: how to destroy a tumour with rays without harming the healthy tissue around it. The answer has the same structure, many weak rays converging from different directions. Told to use the story, most people found it. Given no hint, far fewer did (Gick & Holyoak, 1980. People held the key and did not think to try it.

This is among the most reliable findings in the psychology of thinking, and it bears directly on Part II. The great discoverers were people who noticed, without a hint, that something from one field was the key to a problem in another. Douglas Hofstadter has gone further and argued that analogy is not one mental tool among several but the centre of all thinking, including the simple act of recognising a new thing as one of a familiar kind (Hofstadter & Sander, 2013).

6.2 Categories have a centre, and a natural level

Wittgenstein's remark about games (section 2.9) was tested in the 1970s by Eleanor Rosch. She found that the members of a category that people rate as most typical are the ones sharing the most features with the other members. In one experiment with 400 participants, the agreement between this family-resemblance score and people's ratings of typicality ran between 0.84 and 0.94 (Rosch & Mervis, 1975. A robin is a better bird than a penguin, and everyone agrees. Categories are organised around a centre, not bounded by a definition.

Rosch also found that abstraction has a natural level. “Chair” sits between “furniture” above and “kitchen chair” below. That middle level is the one children name first and adults use most. It is the highest level at which the members still share a shape and a way of being handled (Rosch et al., 1976. Thinking above or below it takes more effort.

How concepts are stored remains unsettled. Lawrence Barsalou has argued that a concept is a partial re-running of the sensations and actions that went with the real thing (Barsalou, 1999, 2008). Critics reply that activity in the sensory parts of the brain during thought does not prove that concepts *are* sensory, and that at some level concepts must be abstract symbols (Mahon & Caramazza, 2008). The hub-and-spoke account of section 5.3 lies between the two. It is Locke against Berkeley again, argued now with brain scanners.

6.3 Metaphor, models and distance

Three further ideas complete the picture.

George Lakoff and Mark Johnson proposed in 1980 that we grasp abstract matters through bodily ones, and that this shapes thought and not only speech. Good is up and bad is down. An important matter is a weighty one (Lakoff & Johnson, 1980). The experimental support for the strong version of the claim is mixed. As a description of how people talk their way into abstract subjects it is hard to dispute.

Philip Johnson-Laird’s theory holds that people reason by building small models of the situation in the mind, one for each possibility, and reading the conclusions off them. To save effort, the models show what is true and leave out what is false. The economy has a price: it produces illusions of reasoning so compelling that in one problem about 95% of people were wrong. Problems that need more than about three models are hard for everyone (Johnson-Laird, 2010).

Yaacov Trope and Nira Liberman showed that distance itself makes thought abstract. Whatever is far away, in time, in space, socially or in likelihood, is thought of in terms of its purpose and gist. Whatever is near is thought of in its details. Next year’s holiday is “a rest”. Tomorrow’s is tickets and luggage (Trope & Liberman, 2010). The practical use is plain: to think more abstractly about a problem, step back from it.

6.4 Expertise is better abstraction, not better memory

A chess master, shown a position from a real game for five seconds, can set it up again almost perfectly. In the classic study, the master placed about 16 pieces correctly, a strong club player about 8 and a beginner about 4. When the pieces were scattered at random, the master’s advantage vanished (Chase & Simon, 1973). The master’s memory is not larger. The master sees the board in bigger units, as familiar formations, where the beginner sees single pieces.

The same holds in science. Asked to sort physics problems into groups, beginners sorted by what the problems looked like: those with ramps together, those with pulleys together. Experts sorted by the principle needed to solve them, such as conservation of energy, although nothing on the surface of the problems said so (Chi et al., 1981). The expert has stopped looking at the surface.

6.5 Learning a concept from a single example

Show a person one example of an unfamiliar written character and they can recognise it again, copy it, and invent others in the same style. In 2015 a computer model matched this ability by treating each character as a small programme, built from strokes in a particular order. Asked to identify a character from one example among twenty, the model was wrong 3.3% of the time and people 4.5%. A standard image-recognition network of that period was wrong 13.5% of the time (Lake et al., 2015).

The interest lies in what the model had to be given: the knowledge that characters are composed of parts, that they are produced by a process, and that what is learned from old cases helps with new ones.

Remove any of these and its performance fell. People learn abstractions quickly because they come to each new one with a stock of earlier ones.

Next: what injury, illness, age, stress and lost sleep do to this ability, and three widely held beliefs about it that the evidence does not support.

7. When it fails

In brief: abstract thinking can be damaged, and the clinic has corrected several mistaken beliefs about who can and cannot do it.

The clinic has taught as much as the laboratory, and it has corrected several mistaken beliefs.

Damage to the frontal lobes. A study of patients with frontal injuries found that their poor scores on classic tests such as card sorting were entirely accounted for by a drop in fluid intelligence, wherever in the frontal lobe the damage was. Certain other difficulties, with proverbs, with juggling several tasks, and with reading a social situation, were not accounted for in that way, and went with damage at the very front on the right (Roca et al., 2010). The general capacity is lost first, and some of the most abstract skills have their own vulnerability.

Schizophrenia. For a century, concrete thinking was taken to be a mark of schizophrenia, and the proverb test was built on that belief. A 2024 review of 27 studies revised it. People with schizophrenia do give more concrete answers than healthy people. So do people with other conditions, mainly depression, and to a similar degree. The reviewers concluded that “concreteness seems transdiagnostic” (Felsenheimer & Rapp, 2024). The sign is real and is not specific to one illness.

Autism. A common stereotype holds that autistic people cannot think abstractly. The evidence is against it. In one study, 38 autistic children scored on average about 30 percentile points higher on Raven’s test, a demanding measure of abstract reasoning, than on a standard intelligence scale, and some scored more than 70 points higher (Dawson et al., 2007). A review of more than 50 studies concluded that what distinguishes autistic perception is a strong eye for detail, which is better described as a style than as a deficit (Happé & Frith, 2006).

Age. Reasoning on unfamiliar problems is at its best in early adulthood and declines gradually after that, and one large study places the start of the decline in a person’s twenties or thirties. Knowledge moves the other way and goes on growing until at least sixty (Salthouse, 2009). When the decline begins is disputed. The pattern is Cattell’s two kinds of intelligence seen across a lifetime, and it offers a consolation: what is lost in quickness is repaid in the stock of abstractions already held.

Stress and lost sleep. The prefrontal cortex is the part of the brain most easily upset by its own chemistry. Even mild stress that a person cannot control causes a rapid loss of its functions (Arnsten, 2009). Loss of sleep spares routine work and impairs exactly the thinking that deals with the unexpected: innovation, the revising of plans, and flexibility (Harrison & Horne, 2000). > **The conditions under which people are most often asked to think creatively, under pressure and short of sleep, are the ones in which the brain is least able to do so.**

Next: the question that matters most. Can a person become better at this, and what actually works?

8. Can it be grown?

In brief: yes. Sustained learning raises it, across a lifetime and across whole populations. Shortcuts do not. What a person brings to it matters as much as what they are given.

> **“Anyone can expand their intelligence. They need the willpower and determination to succeed, with a passion for knowledge.”** Maynard L. Dokken

8.1 What works: education

In the early 1930s Alexander Luria, a student of Vygotsky, travelled to remote parts of Central Asia and put simple reasoning problems to farmers who had never been to school and to their neighbours who had had a little schooling. One problem ran: “In the far north, where there is snow, all the bears are white.” Novaya Zemlya is in the far north. What colour are the bears there? Many of the unschooled adults declined to answer. They had not been there, they said, and they spoke only of what they had seen. To those with a year or two of school the answer was obvious (Cole, 2017).

The study could not be repeated, and Michael Cole, who edited its English edition, cautions that the farmers may simply have been unwilling to play an unfamiliar game. Later research showed that unschooled adults do reason from the words alone when a problem is presented as a story. The central observation has held. Reasoning from a premise one has no experience of is a learned habit, and schooling is where it is learned.

The modern evidence is strong. An analysis of 42 sets of data covering more than 600,000 people, using studies designed to separate cause from mere association, found that each additional year of education raises measured intelligence by between one and five points, and that the gain lasts through life. The authors call education the most consistent and durable method yet found for raising intelligence (Ritchie & Tucker-Drob, 2018). The Flynn effect of section 3.4 is the same thing seen across whole nations.

8.2 What does not work: shortcuts

In 2008 a study reported that practising a demanding memory game raised scores on an unrelated test of fluid reasoning (Jaeggi et al., 2008). An industry of brain training followed. The larger reviews that came afterwards did not support it. People improve at the games they practise. Against proper comparison groups, the improvement does not carry over to intelligence, reading or arithmetic (Melby-Lervåg et al., 2016). A review of the studies cited by the companies themselves reached the same conclusion (Simons et al., 2016).

The same holds for two popular beliefs about children. Music lessons have many merits, but a pooled analysis found only a small effect on thinking skills, and one that shrank as the quality of the studies rose (Sala & Gobet, 2017). No comparable evidence was found that studying mathematics by itself improves reasoning in general.

There is one more encouraging result, small and in need of repeating. Children who played games chosen to exercise reasoning for eight weeks gained about ten points on a test of non-verbal reasoning, while children who played games of speed became faster instead (Mackey et al., 2011).

8.3 Practice, and its limits

The belief that ten thousand hours of practice produces mastery comes from a 1993 study of violinists by Anders Ericsson and colleagues, who argued that expert performance reflects long, effortful, deliberate practice (Ericsson et al., 1993). The idea was later made into a rule by popular writers. A pooled analysis of 88 studies found that deliberate practice accounts for about a quarter of the differences between people in games, about a fifth in music, and under one per cent in the professions (Macnamara et al., 2014). When the violin study itself was repeated with tighter methods, the best players had not practised more than the merely good ones (Macnamara & Maitra, 2019).

Practice is necessary. It is what builds the large units of section 6.4, and no one in Part II did without it. It is not the whole explanation of exceptional achievement.

8.4 What stands in the way

If abstract thinking can be grown, the question is why it so often is not. Part of the answer lies in the evidence above: it takes sustained effort, and there is no shortcut. Part of it lies outside the individual. A school that rewards the memorised answer over the question, and that treats each new tool of thought as a threat to be kept out of the classroom, trains people away from the very habits this article describes. The author's first venture, described in Part V, grew out of frustration with exactly that. The pattern is old. Writing was feared, and so was the calculator, and Part IV shows that the fear has a new object now. People turn away from what they do not yet understand, and the cost is paid by those who are never shown what their minds can do.

The author's own conviction, stated in the words at the head of this section, goes beyond what the studies can prove. They show that education and effort raise ability. They do not show that there is no limit for any person. They do not show that there is one.

8.5 What the evidence supports

Put together, the findings of Part I point to a short list of things that do strengthen abstract thinking.

- **Sustained education**, in which a person is repeatedly asked to reason from premises, to classify, and to take hypothetical questions seriously.
- **Comparison.** Infants abstract a relation by comparing several examples, and students do the same. Two cases set side by side teach more than either alone.
- **Deep knowledge of a field**, which turns surfaces into principles.
- **Notation and other outside aids**: a diagram, an equation, a written plan. Section 9 explains why these matter so much.
- **Distance**: stepping back in time or in viewpoint.
- **Sleep, after working on the problem.**
- **A calm mind.**

Next: the measured limits of the mind, which are startlingly small, and the reason they have never stopped anyone.

9. The real limits, and why "limitless" is still the right word

In brief: a mind can hold about four things at once. Abstraction is how so small a workshop takes in so much, and it is why there is no highest idea.

9.1 A very small workshop

In 1956 George Miller published one of the best-known papers in psychology, "The magical number seven, plus or minus two". People can hold about seven items in mind at once, he reported, and the important point was what counts as an item. The limit is on *chunks*, meaningful units, and not on the quantity of information in them. Seven letters or seven words or seven familiar phrases are all about equally easy. So a person can hold more by recoding information into larger units, and Miller regarded that recoding as central to thought (Miller, 1956).

Later work lowered the number. When people are prevented from rehearsing and regrouping, Nelson Cowan showed, the true capacity is about four chunks (Cowan, 2001).

Graeme Halford and colleagues measured the limit that matters most for abstraction: how many things can be related to one another at the same moment. The answer is again about four. When people were asked to interpret graphs in which several factors interacted, their performance fell sharply between three factors and four. With five, they did no better than chance (Halford et al., 2005).

> **The mind that produced general relativity can relate about four things at once.**

9.2 How a small workshop builds without limit

The same researchers explain how the limit is overcome, and there are only two ways. One is to break a problem into steps, each small enough to fit. The other is to collapse several things into one: to form a chunk (Halford et al., 1998).

To form a chunk is to abstract. When Peirce's operation turns "moves faster and faster" into "acceleration", four things have become one, and there is room for three more. When a chess master sees a formation instead of five pieces, the same has happened. Every abstraction is a way of fitting more of the world into a workshop that does not get any bigger.

And the chunks can be kept outside the head. A written symbol holds an idea in place while the mind works on another. A diagram shows relations so that they need not be remembered. An equation carries out, on paper, steps that no mind could hold together. The history of thought is in large part a history of such aids: number systems, algebra, the coordinate graph, the notation of the calculus, the musical score, the computer program. Each one let the next generation begin where the last one stopped.

> **"By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems."** Alfred North Whitehead (Whitehead, 1911)

Each was also resisted when it was new, a point Part IV returns to.

This is the sense in which the mind of abstract thinking is limitless. The workshop is small and fixed. What can be brought into it has no fixed limit, because any structure, once understood, can be folded into a single unit and used as a building block for the next. There is no highest abstraction.

9.3 What is still not known

A candid account should end with the open questions. It is not settled how the brain represents an abstract concept: as a re-run of experience, as a symbol, or as both joined at a hub. It is not settled whether the frontal lobes form a strict hierarchy. It is not known whether the grid-like signals seen for ideas come from the same cells that map space. No one has explained why the rise in test scores has slowed, and in some countries reversed. It is not known whether any other animal reasons about relations between relations. And it is not known whether training in reasoning produces gains that last and that reach into ordinary life.

What is known is enough for the rest of this article. Abstract thinking is a definite set of abilities. It is present in a simple form before speech, it is built up by teaching and by culture, it depends on identifiable systems in the brain, and it is extended without limit by chunking and by notation. Part II turns to the people who extended it furthest.

Next: twenty-five minds, and the exact step each one took that no one had taken before.

Jump to: Contents · Part I · Part II · Find a mind · Part III · Part IV · Part V · Closing

Part II. Twenty-five minds

Part I described a mental operation. Part II is the record of what it has done. The twenty-five people below are set out in order of time. Each was chosen because the historical record shows a definite abstract move, a step away from the particular case, and a discovery or invention that followed from it.

Three remarks about the selection.

First, it is a list of discoverers and inventors, as the title of this part of the article requires. The artists, composers and writers who did the same thing in their own fields are in Part III.

Second, any such list is a judgement. Another author could defend a different twenty-five, and several who have an equal claim are named at the end of this part.

Third, the stories have been checked. Several of the best-known anecdotes in the history of science turn out to rest on a single source written long afterwards. Where that is so, the profile says so. The discoveries lose nothing by it.

For each mind the profile gives **the abstraction**, in a sentence, and then the account of how it was made and what came of it.

Find a mind

Each name is a link to that person's profile.

Field	Minds
Language and logic	Panini, Aristotle, Kurt Gödel
Mathematics	Euclid, Archimedes, Brahmagupta, Al-Khwarizmi, René Descartes, Gottfried Wilhelm Leibniz, Leonhard Euler, Carl Friedrich Gauss, Georg Cantor, Emmy Noether
Physics and astronomy	Galileo Galilei, Johannes Kepler, Isaac Newton, Michael Faraday, James Clerk Maxwell, Marie Curie, Albert Einstein
Life and matter	Charles Darwin, Gregor Mendel, Dmitri Mendeleev
Computing and information	Alan Turing, Claude Shannon
And the author	The thinking at work, in eight ventures

1. Panini (India, about the 5th century BCE)

The abstraction: a whole language described as a finite system of rules.

About two and a half thousand years ago a scholar named Panini wrote a description of Sanskrit unlike anything before it. His *Ashtadhyayi*, "eight chapters", sets out the language in just under four thousand rules, each of them a few syllables long (Encyclopaedia Britannica, n.d.-n; Kiparsky, n.d.).

To make that possible he had to invent a language for talking about language. He attached markers to roots and endings that carry grammatical information but are never pronounced. He named whole classes of sounds by short codes. He laid down that rules apply in a strict order, and that a specific rule overrides a general one, a principle that modern linguistics later adopted. He even treated the dropping of a sound as its replacement by a "zero" element (Kiparsky, n.d.).

The linguist Paul Kiparsky calls the result "the most complete generative grammar of any language yet written" (Kiparsky, n.d.). Panini had seen that the endless variety of what can be said is produced by a small set of rules applied again and again. The same insight, that a finite set of rules can generate unlimited output, would not be reached in Europe for more than two thousand years. It is one of the foundations of modern linguistics.

2. Aristotle (Greece, 384–322 BCE)

The abstraction: an argument is valid because of its form, whatever it is about.

Aristotle studied in Plato's Academy for twenty years and then founded his own school in Athens. Among a vast body of work, one achievement stands apart. He noticed that reasoning has a shape that can be studied independently of its subject. If one thing belongs to all of a second, and the second belongs to all of a third, then the first belongs to all of the third. It does not matter whether the things are animals, triangles or cities (Smith, 2022).

Having seen this, he did what no one had done: he went through the possible patterns of premises systematically, showed which ones always yield a conclusion, and proved that the rest do not. The result, in the *Prior Analytics*, is the earliest formal study of logic that we possess (Smith, 2022).

Its influence is hard to overstate. For two thousand years Aristotle's logic *was* logic, and Kant believed there was nothing left to add. Frege showed in 1879 that there was (Part I, section 2.6), and modern logic, on which every computer rests, starts there. But Frege and his successors were doing what Aristotle had shown could be done. The idea that thinking itself has a structure which can be written down, examined and checked is his.

3. Euclid (Alexandria, about 300 BCE)

The abstraction: all of geometry derived from a handful of stated assumptions.

Almost nothing is known of Euclid's life except that he taught in Alexandria. His *Elements* is another matter. In thirteen books it gathers the mathematics of its age, plane and solid geometry and the theory of numbers, and arranges it in a new way. It begins with definitions, five postulates and a few "common notions", and everything that follows is proved from those and from what has already been proved (O'Connor & Robertson, n.d.-f).

Euclid probably did not discover most of the results. His contribution was the system. He turned a collection of facts about shapes into a single structure in which each truth rests on the ones beneath it, down to a foundation small enough to be inspected.

The *Elements* was first printed in 1482 and has gone through more than a thousand editions. For some two thousand years it was the chief source of geometric reasoning and the model of what it means to prove something (O'Connor & Robertson, n.d.-f). The system was not perfect. Unstated assumptions were noticed even in antiquity, and when mathematicians in the nineteenth century asked what would happen if one postulate were changed, they found new geometries. That, too, was only possible because Euclid had written the assumptions down.

The remark that there is "no royal road to geometry", which he is supposed to have made to a king, comes from a writer seven and a half centuries later who introduces it with the words "they say" (O'Connor & Robertson, n.d.-f).

4. Archimedes (Syracuse, 287–212 BCE)

The abstraction: geometric figures treated as objects with weight, and balanced on an imaginary lever.

Archimedes found the area under a curve and the volume of a sphere eighteen centuries before the calculus. How he did it was unknown until 1906, when a lost work, *The Method*, was recovered from a tenth-century manuscript. In it he explains his procedure in a letter to a friend. He imagined a figure cut into very thin slices, treated each slice as if it had weight, and balanced the slices against those of a figure he already understood, as on the arm of a lever. The balance gave him the answer. He then proved it by

strict geometry, because he did not regard the mechanical argument as a proof (MacTutor, n.d.; O'Connor & Robertson, n.d.-b).

> **"Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards."** Archimedes, *The Method* (MacTutor, n.d.)

This is one of the clearest statements anywhere of how discovery and proof differ, and of how an abstraction from one field, the physics of balance, can solve a problem in another, the measurement of shapes. Archimedes also worked out the laws of floating bodies and the principle of the lever, and trapped the value of π between two fractions by reasoning about polygons of 96 sides (O'Connor & Robertson, n.d.-b).

The story that he leapt from his bath shouting "Eureka" is first told by the Roman writer Vitruvius, about two hundred years after his death. Archimedes himself never mentions it, and Galileo pointed out that the method described would not have been accurate enough to work (Scientific American, n.d.; Vitruvius, n.d.). His real discovery about floating bodies is in his own surviving book, and it is greater than the story.

5. Brahmagupta (India, 598–670)

The abstraction: nothing, and less than nothing, treated as numbers.

A sign to mark an empty place in a written number is old. The Babylonians had one, and so did the Maya (O'Connor & Robertson, 2000). The step that Brahmagupta took in 628, in a work written when he was thirty, was of a different order. He treated zero as a number, and negative quantities as numbers, and gave the rules for calculating with them.

He called positive numbers "fortunes" and negative numbers "debts", and he stated what happens in every combination. A debt minus zero is a debt. The product of two debts is a fortune (O'Connor & Robertson, n.d.-c). The Bodleian Library describes his book as the first to discuss zero as a number in its own right (Bodleian Libraries, University of Oxford, 2017).

It is hard now to feel how strange this was. A number had always been a number *of* something. Brahmagupta asked the reader to count what is not there, and what is owed, on the same footing as what is present. He did not get everything right. He held that zero divided by zero is zero. But the move itself, giving a name and rules to an absence, is the model for many later ones: the negative number, the imaginary number, the empty set.

Without such rules there is no algebra as we know it, and no arithmetic that a machine can perform.

6. Al-Khwarizmi (Baghdad, about 780–850)

The abstraction: the method, and not the single problem, as the thing to be studied.

Muhammad ibn Musa al-Khwarizmi worked in Baghdad at the House of Wisdom under the Caliph al-Mamun. Mathematicians before him had solved problems involving unknown quantities one at a time, each with its own ingenuity. Al-Khwarizmi showed that every such problem, up to those involving the square of the unknown, can be reduced to one of six standard forms, and he gave a general procedure for each (O'Connor & Robertson, n.d.-a).

His book's title contained the word *al-jabr*, "completion", meaning the moving of a subtracted quantity to the other side of an equation. From it we have the word algebra. He wrote entirely in words, with geometric arguments in place of symbols (Encyclopaedia Britannica, n.d.-a). The symbols came later. The idea was already complete: one does not solve a problem, one solves a *kind* of problem, once, for good.

A second book explained the Indian system of numerals, including Brahmagupta's zero, and the rules for calculating with them. It survives only in a Latin translation, which begins with a Latin form of the author's name. From that name comes the word *algorithm* (O'Connor & Robertson, n.d.-a).

Two of the central words of the modern world, then, come from one man in ninth-century Baghdad. Both name the same abstraction: a procedure that can be stated in general and carried out without further thought. Every computer program is one.

7. Galileo Galilei (Italy, 1564–1642)

The abstraction: motion as it would be if nothing interfered.

For two thousand years the accepted view, Aristotle's, was that heavier bodies fall faster. It agrees with what we see: a stone beats a feather. Galileo's achievement was to see that what we see is the wrong place to start.

He argued first by pure thought. Suppose a heavy stone falls faster than a light one, and tie the two together. The light stone should hold the heavy one back, so the pair falls more slowly. But the pair is heavier than either, so it should fall faster. The assumption contradicts itself (Galilei, 1914).

Then he asked what the real law is, and to find it he removed, in imagination, everything that complicates a real fall. His conclusion, in the words of Salviati, his spokesman in *Two New Sciences*: "in a medium totally devoid of resistance all bodies would fall with the same speed" (Galilei, 1914) (p. 72). No one had ever seen such a medium. He slowed the motion down by rolling balls along sloping planes, and found that the distance covered grows as the square of the time (O'Connor & Robertson, n.d.-g).

> **The friction and the air are real. He treated them as disturbances laid over a simple law.**

This is idealisation, and Galileo is its founder. The friction and the air are real. He treated them as disturbances laid over a simple law, and stated the law for the case in which they are absent. Every branch of physics has proceeded in this way since. He also said plainly what language the simple laws are written in: the book of nature, he wrote, is written in the language of mathematics (O'Connor & Robertson, n.d.-g).

That he dropped weights from the Leaning Tower of Pisa was claimed later by his pupil and biographer, and is not otherwise recorded. The defiant words he is said to have muttered after his trial are not found in the reliable sources.

8. Johannes Kepler (Germany, 1571–1630)

The abstraction: exact mathematical law drawn out of a mass of observations.

Kepler inherited the best observations of the planets ever made, those of Tycho Brahe, and set himself to find the orbit of Mars. He shared the assumption of every astronomer since antiquity, that the heavenly bodies move in circles. After years of calculation his best circular orbit still disagreed with Tycho's observations, by eight minutes of arc, about a quarter of the width of the full Moon.

Almost anyone would have let so small an error pass. Kepler would not. He later wrote that those eight minutes had led the way to the reformation of all astronomy (Di Liscia, 2025). He gave up the circle and found that the orbit is an ellipse with the Sun at one focus. He found that a planet sweeps out equal areas in equal times, and, ten years later, that the square of the time each planet takes to go round is proportional to the cube of its distance from the Sun (O'Connor & Robertson, n.d.-l).

Two things make Kepler a model. He respected the data absolutely, to the point of abandoning a belief two thousand years old. And he was not content with a description. He wanted the *cause*, and looked for a

force from the Sun that drives the planets, by analogy with magnetism (Di Liscia, 2025). The cause he proposed was wrong. The demand was right, and Newton answered it, using Kepler's third law to do so.

It should be added that Kepler's first great idea, that the spacing of the planets follows from the five regular solids nested inside one another, was mistaken. He never gave it up. A wrong abstraction, pursued with complete seriousness against exact data, led him to the right ones.

9. René Descartes (France, 1596–1650)

The abstraction: a curve is an equation, and an equation is a curve.

Before Descartes, geometry and algebra were separate subjects. Geometry dealt with figures and algebra with unknown quantities. In 1637, in an appendix to his *Discourse on Method*, Descartes joined them. A curve, he showed, can be described by an equation relating two varying lengths, and a problem about figures can therefore be turned into a problem about equations and solved by calculation (Domski, 2025).

To do it he had to change what algebra meant. For the Greeks, the product of two lengths was an area and of three a volume, so that an expression mixing them made no sense. Descartes treated every product as simply another length. All the terms of an equation then become things of the same kind, free to be combined (Domski, 2025). He also introduced the notation still in use: letters from the start of the alphabet for known quantities, x, y and z for unknown ones, and raised numerals for powers.

The consequences were immediate and vast. Shapes could be calculated with. Motion, which traces a curve, could be written as an equation. Within fifty years Newton and Leibniz had built the calculus on this ground, and every graph ever drawn is a descendant.

The pleasant story that Descartes had the idea while watching a fly on his ceiling does not appear in the scholarly accounts consulted. His own axes, incidentally, were not the pair at right angles that now bears his name.

10. Isaac Newton (England, 1643–1727)

The abstraction: one law for the Earth and the heavens.

Since antiquity it had been taken for granted that the heavens and the Earth are different realms under different rules. On Earth, things fall. In the sky, they circle. Newton's greatest act was to deny the distinction. The force that pulls a stone to the ground and the force that holds the Moon in its path are the same force, and it acts between every piece of matter in the universe and every other (O'Connor & Robertson, n.d.-j).

He made the connection visible with an image. Fire a cannonball horizontally from a mountaintop. It lands some way off. Fire it faster and it lands further. Fire it fast enough and the ground curves away beneath it as quickly as it falls, and it never lands at all. An orbit is a fall that keeps missing (Brown & Fehige, n.d.). The Moon is falling.

To turn the idea into a calculation he needed mathematics that did not yet exist, a way of handling quantities that change continuously, and he created it during two years when plague had closed his university. He published the whole system in the *Principia* in 1687 (O'Connor & Robertson, n.d.-j).

The fall of an apple did play a part, on Newton's own word. He told the story late in life to a friend, who recorded that the idea of gravitation was occasioned by the fall of an apple as Newton sat in a contemplative mood. In that account nothing strikes him on the head (Matson, 2010).

Newton is the supreme case of unification: seeing that two things everyone held to be different are one. His well-known line, "if I have seen further, it is by standing on the shoulders of giants", is genuine. It

comes from a letter to Robert Hooke dated February 1675 in the calendar then in use (Historical Society of Pennsylvania, n.d.), and the giants are in this list.

11. Gottfried Wilhelm Leibniz (Germany, 1646–1716)

The abstraction: a notation that does the thinking.

Leibniz arrived at the calculus independently of Newton and a few years later, and the two men's followers fought bitterly over the credit. What matters here is the difference in how they wrote it. Leibniz gave long thought to his symbols. He first used the long *S* for a sum, the integral sign, in a manuscript of November 1675, and with it the *d* for a small difference (O'Connor & Robertson, n.d.-i).

The symbols were designed so that the rules of the subject follow from the way they are written. A student who manipulates them correctly gets the right answer almost without needing to understand why. It is Leibniz's notation, not Newton's, that the whole world uses today.

This was not an accident of taste. Leibniz held a conviction that ran through all his work: that a well-chosen system of signs could carry the burden of reasoning, and that it might one day be possible to settle disputes by calculation. He sketched such a universal language of thought as a young man and returned to it throughout his life (Look, 2013). He built a calculating machine that he showed to the Royal Society in 1673. And he worked out, by 1679, that every number can be written using only the digits 0 and 1 (Norman, n.d.-b; O'Connor & Robertson, n.d.-i).

He did not foresee the computer, and the binary system was for him more a philosophical curiosity than a practical tool. But the two ideas that the computer joins, that reasoning can be mechanised and that all information can be coded in two symbols, are both in his papers.

Leibniz shows something that Part I arrived at from another direction (section 9.2). A good notation is not a way of recording thought. It is a tool that extends what a mind can do.

12. Leonhard Euler (Switzerland and Russia, 1707–1783)

The abstraction: a city reduced to what is connected to what.

The city of Königsberg had seven bridges linking two islands and the banks of its river. Its citizens wondered whether one could take a walk that crossed each bridge exactly once. In 1735 the problem reached Euler, the most productive mathematician of his century, who noted that it belonged to no existing branch of mathematics, since it involved neither measuring nor calculating (Paoletti, n.d.).

His solution was to throw nearly everything away. The lengths of the bridges, the shape of the islands and the distances between them are all irrelevant. All that matters is which pieces of land are joined to which, and by how many bridges. Seen that way, the question has a simple answer. Whether such a walk exists depends only on how many bridges meet at each piece of land, and in Königsberg it does not (Euler Archive, n.d.).

With that paper Euler began two new subjects, the study of networks and the study of those properties of shape that survive stretching. A map of an underground railway, which shows the connections and ignores the distances, uses the same abstraction, and so does the analysis of any network.

Euler also gave mathematics much of its language: the notation *f(x)* for a function, *e* for the base of natural logarithms, *i* for the square root of minus one, and the sign Σ for a sum. He went blind in his sixties and produced nearly half of his enormous output afterwards, working in his head and dictating (O'Connor & Robertson, n.d.-n).

13. Carl Friedrich Gauss (Germany, 1777–1855)

The abstraction: the curvature a surface has in itself.

We think of a curved surface as curved *in space*: we see that a ball is round by looking at it from outside. In 1828 Gauss proved something that seems at first impossible. The curvature of a surface can be found entirely from measurements made within the surface, of lengths and angles, without any reference to the space around it (O'Connor & Robertson, n.d.-k). Creatures living in a surface, unable to leave it or look at it from outside, could still discover whether their world was flat.

The result changed the question of what geometry is about. If curvature belongs to a surface itself, then it makes sense to ask whether *space* is curved, and to settle the matter by measurement. Gauss privately explored geometries that break Euclid's rules and kept silent, telling a friend he feared for his reputation (O'Connor & Robertson, n.d.-k). Bernhard Riemann extended the idea to spaces of any number of dimensions, and Einstein later used it to describe gravity.

Gauss is on this list as one of the most powerful mathematical minds on record. As a young man he showed how to construct a regular figure of seventeen sides. At twenty-four he published the book that founded the modern theory of numbers, and in the same year he predicted, from a few weeks of observations, where a lost minor planet would reappear (O'Connor & Robertson, n.d.-k).

14. Michael Faraday (England, 1791–1867)

The abstraction: the space around a magnet is not empty.

Faraday had only a basic schooling and was apprenticed to a bookbinder at fourteen. He knew almost no mathematics (Encyclopaedia Britannica, n.d.-m; O'Connor & Robertson, n.d.-o). He became the greatest experimental scientist of his century and, by a different road from everyone else on this list, one of its great abstract thinkers.

The accepted view of his day, following Newton, was that electric and magnetic bodies act on one another directly across empty space. Faraday pictured something else. Sprinkle iron filings round a magnet and they arrange themselves in curves. He took the curves seriously. He imagined the space around every magnet and every current as filled with *lines of force*, and he held that these lines were physically real, a state of strain in space itself (Encyclopaedia Britannica, n.d.-m).

The picture was not a decoration. It told him what to look for. In 1831 he found that a current is produced in a wire when the magnetism around it changes, and he understood the effect in his own terms: the current depends on how many lines of force the wire cuts in a given time. Every electrical generator works on this principle (Encyclopaedia Britannica, n.d.-m).

Trained physicists were slow to follow him, since he offered no equations. James Clerk Maxwell saw what he had done.

> **"Faraday, in his mind's eye, saw lines of force traversing all space where the mathematicians saw centres of force attracting at a distance."** James Clerk Maxwell (Maxwell, 1873)

Maxwell insisted that Faraday's way of conceiving things was itself mathematical, and he set out to put it into symbols. The result is described in profile 17.

Faraday is the proof that abstract thinking does not require mathematics. It requires seeing a structure that the senses do not show. The idea of a *field* that he introduced is now the foundation of all physics.

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15. Charles Darwin (England, 1809–1882)

The abstraction: what a breeder does by choice, nature does without one.

Darwin came back from the voyage of the *Beagle* convinced that species change, and without any idea of how. The route to his answer is preserved in his notebooks, and it was not a flash. It took years and ran through two borrowed ideas.

The first came from the farmyard. Breeders of pigeons and cattle produce new varieties by picking out, generation after generation, the individuals with small differences they happen to want. Darwin wrote in his autobiography: “I soon perceived that selection was the keystone of man’s success in making useful races of animals and plants” (Darwin, 1958). But in nature there is no breeder. What does the selecting?

The second came in the autumn of 1838, when he read, by his own account for amusement, Thomas Malthus’s essay on human population. Malthus argued that people multiply faster than their food supply, so that many must perish. Darwin saw that this is true of every living thing. Far more are born than can survive. Those that differ in some way that helps them will survive more often and leave more offspring, and the helpful difference will spread. “Here, then, I had at last got a theory by which to work” (Darwin Correspondence Project, 2015; Darwin, 1958).

That is the whole theory, and its structure is worth noticing. It is an analogy, from the breeder to nature, completed by a generalisation, from Malthus’s humans to all life. The result is a mechanism with no one operating it, which nevertheless produces the appearance of design (Darwin Correspondence Project, 2016).

Darwin sketched the theory in 1842, told no one for two years, and published only in 1859, after Alfred Russel Wallace had arrived independently at the same idea and the two had been announced together (Darwin Correspondence Project, 2015; Darwin Correspondence Project, 2016). The popular story that the finches of the Galápagos gave him the idea in a moment has no support in the record. The documented path is slow, and it is made of notebooks.

16. Gregor Mendel (Moravia, 1822–1884)

The abstraction: inheritance as the combination of unseen units, written as letters.

While Darwin was preparing his book, an Augustinian friar in the city of Brünn was growing peas in the monastery garden. He approached heredity in a way no one had: he simplified it and then he counted.

He ignored nearly everything about his plants and followed one sharply contrasting trait at a time, such as the shape of the seed. He began with lines that bred true. He crossed them, crossed the offspring, and counted the results in their thousands. In the second generation the two forms appeared in “an average ratio of 2.98:1, or 3:1” (Mendel, 1866).

His explanation was a piece of algebra. Suppose each plant carries two hidden factors for the trait, one from each parent. Write the dominant one as A and the other as a. A hybrid is Aa. When hybrids are crossed the factors combine by chance, giving A, Aa, Aa and a in equal numbers, which is one, two and one. Since A masks a, three plants in four show the dominant form (Mendel, 1866).

No one had seen these factors, and no one would for decades. Mendel inferred discrete, invisible units of heredity from nothing but ratios, and he represented them with letters. *Encyclopaedia Britannica* calls him the first to lay the mathematical foundation of genetics (Encyclopaedia Britannica, n.d.-h).

He presented the work in 1865 and published it the next year. It was ignored until 1900, sixteen years after his death, when three botanists published results that confirmed it (Fairbanks & Abbott, 2016). A later charge that his figures were too good to be honest was examined at length and is regarded as settled in his favour (Fairbanks & Abbott, 2016).

17. James Clerk Maxwell (Scotland, 1831–1879)

The abstraction: a model built in order to be taken away.

Maxwell set himself the task that Faraday had left: to put the lines of force into mathematics. The way he did it is one of the most instructive episodes in the history of thought, because he said at each step exactly what he was doing.

He began with deliberate analogy. “By a physical analogy,” he wrote, “I mean that partial similarity between the laws of one science and those of another” which allows each to illustrate the other (Tripodi & Ruta, 2024). He pictured the lines of force as fine tubes carrying an imaginary fluid, borrowed the mathematics of fluid flow, and warned the reader that his picture of Faraday’s “electro-tonic state” involved “no physical theory” and was “only a kind of artificial notation” (Bain, n.d.).

Next he built an elaborate mechanical model of space, filled with spinning cells separated by small idle wheels. He did not claim it was true. It stood to the real mechanism, he said, as an orrery stands to the solar system (Bain, n.d.). But the model produced something. From it he calculated how fast a disturbance would travel through such a medium, and the answer was, very nearly, the measured speed of light. “We can scarcely avoid the inference,” he wrote, “that light consists in the transverse undulations of the same medium” (Bain, n.d.).

Then came the decisive step. In his paper of 1864 he set the wheels and cells aside and presented the equations alone. The mechanical picture had been scaffolding (Bain, n.d.; Tripodi & Ruta, 2024).

The result joined electricity, magnetism and light in one theory. Einstein judged the change in our conception of reality that Maxwell brought about to be the most profound and fruitful since Newton (Encyclopaedia Britannica, n.d.-j).

Maxwell shows the full cycle of abstract thought in a single career: the picture, the analogy, the model, and at last the bare mathematical relations.

18. Dmitri Mendeleev (Russia, 1834–1907)

The abstraction: a pattern trusted more than the data.

By the 1860s chemists knew of dozens of elements and had measured their weights and properties. Several had noticed that certain elements resemble one another. Mendeleev, writing a textbook and needing a way to organise its second volume, found the principle that tied them together. Arranged in order of atomic weight, the elements show their properties recurring at regular intervals. He announced this periodic law in March 1869 (Encyclopaedia Britannica, n.d.-e).

Others came close, and the German chemist Lothar Meyer drew up a similar table at about the same time (Siegfried, 2019). What sets Mendeleev apart is what he did next. Where the pattern required an element that did not exist, he left a gap. He then predicted, from the position of the gap, the properties of the element that would one day fill it.

He described one such element, calling it eka-aluminium, and gave its atomic weight as about 68. In 1875 a French chemist discovered gallium. It matched, except that its measured density differed from the prediction. Mendeleev wrote to say the measurement must be wrong. The sample was purified and measured again, and Mendeleev was right (Siegfried, 2019; Woods, 2007). Two more of his predicted elements, scandium and germanium, were found by 1886 (Encyclopaedia Britannica, n.d.-e).

That is the difference between a classification and a law. Mendeleev believed in the structure he had found more firmly than in the laboratory figures of his day. The table did not foresee everything: a whole family of elements, the noble gases, was discovered later, and no one had predicted them (Scerri, 2012). Its explanation, in terms of the structure of the atom, came only in the next century.

The tale that the table came to him in a dream is doubtful, and so is the tale that he arranged the elements like a game of cards, since no such cards have been found (History Today, n.d.; Scerri, 2012). He had worked on the problem for about ten years.

19. Georg Cantor (Germany, 1845–1918)

The abstraction: counting without numbers, carried into the infinite.

How does one tell whether two collections are the same size without counting them? Pair them off. If every seat in a hall has one person in it and no one is standing, there are as many people as seats, whatever the number.

Cantor took this simple test, which Frege was using at the same period to define number (Part I, section 2.6), and applied it where no one had dared: to collections that have no end. The results were astonishing. The even numbers can be paired off, one for one, with all the whole numbers, so there are just as many of them, though they are only half. The same is true of all the fractions.

Then, in a proof completed in December 1873, he showed that the points on a line *cannot* be paired off with the whole numbers. However the pairing is attempted, some points are always left over. There are more of them. Infinity is not one thing. There are infinities of different sizes (Ferreirós, 2025; O'Connor & Robertson, n.d.-h).

Cantor went on to build an arithmetic of these infinite numbers. He met fierce opposition. The influential mathematician Leopold Kronecker rejected the whole enterprise, and Cantor's papers were obstructed (O'Connor & Robertson, n.d.-h). He defended himself on a principle that speaks for every abstract thinker: mathematics, he held, is free to form whatever concepts it likes, so long as they are consistent (Dauben, 1979).

He was vindicated. The theory of sets that grew from his work became the standard foundation on which nearly all of modern mathematics is built (Ferreirós, 2025).

Cantor suffered from severe depression in later life, and it is sometimes said that contemplating the infinite drove him mad. His biographers do not accept this (O'Connor & Robertson, n.d.-h). What he did was take an idea that had been a mystery since the Greeks, the infinite, and by means of one clear definition make it something that can be reasoned about exactly.

20. Marie Curie (Poland and France, 1867–1934)

The abstraction: the rays come from the atom itself.

It had recently been found that uranium gives off invisible rays. The young Marie Curie chose the phenomenon as the subject of her doctorate, and brought to it something the first observers had lacked: exact measurement. The rays make air conduct electricity slightly. Using a sensitive instrument devised by her husband Pierre and his brother, she measured that conductivity and so obtained, for the first time, a number for the strength of the radiation (American Institute of Physics, n.d.).

The numbers led her to a conclusion of great boldness. The strength of the rays depended only on how much uranium a sample contained. It did not matter what the uranium was combined with, or whether it was solid or powdered, hot or cold. The radiation was therefore not a chemical effect. It had to be a property of the uranium atom itself (American Institute of Physics, n.d.). At a time when the atom was held to be indivisible and unchanging, this pointed to something going on *inside* it. She gave the property its name: radioactivity.

A second inference followed. Certain uranium ores were more radioactive than their uranium could account for. Something else must be in them, in quantities too small to see, and far more active. On the strength of that excess alone, the Curies announced two new elements in 1898, polonium and radium. It

then took more than three years of heavy labour, working through tons of ore, to isolate a tenth of a gram of a radium compound and prove it (American Institute of Physics, n.d.).

Marie Curie received the Nobel Prize in Physics in 1903, with Pierre Curie and Henri Becquerel, and the Nobel Prize in Chemistry in 1911, alone. She was the first woman to win a Nobel Prize and remains the only woman to have won in two sciences (Encyclopaedia Britannica, n.d.-l).

Her abstract move is of a different kind from Cantor's or Maxwell's. It is an inference from measurement: from a number that would not change to a property of the atom, and from a number that was too large to a substance no one had seen. It opened the interior of the atom to science.

21. Albert Einstein (Germany, Switzerland and the United States, 1879–1955)

The abstraction: imagined situations, followed to their conclusion.

At sixteen, Einstein asked himself what he would see if he could run alongside a beam of light at its own speed. According to the physics of the time he should see a light wave standing still, and nothing like that was known or seemed possible. He later called this puzzle the germ of the special theory of relativity (Norton, n.d.). Ten years on, in 1905, while working as a clerk in the Swiss patent office, he resolved it by concluding that time and space themselves are measured differently by observers in motion.

Two years later, still at the patent office, came what he called the happiest thought of his life. A person falling freely does not feel their own weight. Inside a closed, falling room there would be no way of telling that gravity existed at all. Gravity and acceleration, he reasoned, are locally the same thing (Isaacson, 2015). From that one imagined scene it follows that light must bend when it passes a massive body (Brown & Fehige, n.d.).

It took him eight more years to find the mathematics. He needed the geometry of curved spaces that had come down from Gauss and Riemann, and he asked an old classmate to help him learn it (Isaacson, 2015). In November 1915 he presented the result: gravity is not a force. It is the curvature of space and time, produced by matter (Isaacson, 2015).

Einstein left an unusually clear account of how he thought. Asked by the mathematician Jacques Hadamard, he replied that words "do not seem to play any role in my mechanism of thought". The elements of his thinking were "more or less clear images" that he could reproduce and combine at will, and "this combinatory play seems to be the essential feature in productive thought". The words came afterwards, with effort (Hadamard, 1954).

Three corrections are in order. Einstein did not fail mathematics at school: shown a newspaper item that said so, he replied that he had mastered calculus before he was fifteen (Isaacson, 2007). His Nobel Prize was awarded for his work on light and not for relativity (NobelPrize.org, n.d.). And the saying about insanity being the doing of the same thing over and over is not his (Quote Investigator, 2017). A line he did say, in an interview in 1929, sums up this article: "Imagination is more important than knowledge" (Quote Investigator, 2013).

22. Emmy Noether (Germany and the United States, 1882–1935)

The abstraction: every symmetry is a conservation law.

Physics has a small number of great conservation laws. Energy is neither created nor destroyed. Neither is momentum. Before 1918 these were known as separate facts, each discovered on its own. Emmy Noether showed where they all come from.

She proved that whenever the laws governing a physical system stay the same under some continuous change, some quantity must be conserved. If the laws are the same today as they were yesterday, energy is conserved. If they are the same here as over there, momentum is conserved (Freiberger & Thomas,

2025; Noether, 1971). The conservation laws are not separate facts about the world. They are consequences of its symmetries.

This is among the deepest ideas in science, and the physics of the twentieth century was largely built with it. The search for new laws became, to a great extent, a search for new symmetries.

Noether then did something comparable for mathematics. Algebra had been the study of calculations with particular kinds of numbers and expressions. She studied instead the general structures that such systems share and proved theorems about the structures themselves. Her paper of 1921 is described as of fundamental importance in the development of modern algebra (O'Connor & Robertson, n.d.-e). It is abstraction applied to mathematics itself.

She did all this against obstruction. As a woman she was refused a university post for years, and received the right to lecture under her own name only in 1919. In 1933 she was dismissed by the Nazi government because she was Jewish, and she spent her last two years teaching in the United States (O'Connor & Robertson, n.d.-e). When she died, Einstein wrote to the *New York Times* in her honour.

> **"Pure mathematics is, in its way, the poetry of logical ideas."** Albert Einstein, on Emmy Noether, 1935 (Einstein, 1935)

23. Kurt Gödel (Austria and the United States, 1906–1978)

The abstraction: mathematics made to speak about itself.

At the start of the twentieth century the leading mathematicians hoped to place the whole of mathematics on a foundation of axioms from which every true statement could be proved, by rules so exact that a machine could follow them. In 1931 a man of twenty-five showed that this cannot be done.

Gödel's method was an abstraction of extraordinary ingenuity. He devised a code that assigns a number to every symbol, every formula and every proof. Once that is done, a statement *about* formulas and proofs, such as "this formula can be proved", becomes a statement about numbers. Arithmetic can then talk about itself. Using the code, Gödel built a statement of arithmetic that in effect says of itself: *I cannot be proved* (Kennedy, 2025).

Consider what follows. If the statement could be proved, it would be false, and the system would have proved a falsehood. So, if the system is sound, the statement cannot be proved, which is exactly what it says. It is therefore true, and unprovable. Any consistent system of the relevant kind contains truths that it cannot prove, and it cannot prove its own consistency (Kennedy, 2025; O'Connor & Robertson, n.d.-m).

The result is often stretched to mean that nothing can be known for certain. It means nothing of the kind. It is a precise statement about formal systems. But it does mark a real boundary, the first that abstract thought discovered within itself. No single system of rules can capture all mathematical truth. The mind can always step outside the system and see something the system cannot reach.

Gödel spent his later career at the Institute for Advanced Study in Princeton, where his daily walking companion was Einstein (Kennedy, 2025). His method of coding statements as numbers was taken up five years later by a young Englishman.

24. Alan Turing (England, 1912–1954)

The abstraction: a person calculating, reduced to bare essentials.

In 1936 the word "computer" meant a person, usually a clerk, who carried out calculations by following rules. Turing, aged twenty-three, asked a question about such people. What, exactly, can be calculated by following rules?

To answer it he had to say precisely what following rules *is*. He described a human being at work on a calculation and removed everything inessential. “Computing is normally done by writing certain symbols on paper,” he observed. Suppose the paper is divided into squares, like a child’s arithmetic book. The person looks at one square at a time, is in one of a limited number of states of mind, and at each step may write a symbol, move to the next square, and change state. That is all (Hodges, 2013; Turing, 1936).

He called the stripped-down clerk a machine, and there his second insight arrived. The rules a particular machine follows can themselves be written down as symbols on the paper. So it is possible, as he put it, “to invent a single machine which can be used to compute any computable sequence” (Turing, 1936). One machine, given the right instructions, can do the work of all the others.

That is the idea of the computer: a general-purpose device whose behaviour is fixed by a programme held in its own memory. It existed in a mathematics paper before a single one had been built. In the same paper Turing proved that there are well-defined problems no such machine can ever solve, and so found the limits of computing at the moment he defined it (Hodges, 2013).

During the war he worked at Bletchley Park on breaking German ciphers. He did not do so alone, and the essential first steps had been taken by Polish mathematicians, who shared their work with the British in 1939 (GCHQ, n.d.). In 1950 he posed the question of whether a machine could think, and proposed a test for it (Hodges, 2013). Part IV of this article is, in a sense, a footnote to that paper.

Turing’s is perhaps the purest case on record of an abstraction becoming a technology. He began with a person and a sheet of squared paper.

25. Claude Shannon (United States, 1916–2001)

The abstraction: information, separated from meaning.

Shannon made two great abstractions, the first while still a student. In his master’s thesis, submitted in 1937, he noticed that the electrical switching circuits used in telephone exchanges behave according to the rules of the algebra of logic devised by George Boole. A closed switch is “true” and an open one is “false”. The design of circuits, until then a craft, became a matter of calculation (Norman, n.d.-a; O’Connor & Robertson, n.d.-d). Every digital device since has been designed in this way.

The second came in 1948. Engineers wanted to know how much could be sent down a telephone line or through a radio channel. To answer, Shannon had to say what the “much” was a quantity of. His answer began by setting something aside. Messages have meaning, he acknowledged, but “these semantic aspects of communication are irrelevant to the engineering problem” (Shannon, 1948). What matters is only that the message actually sent is one chosen out of a set of possible messages. The more possibilities there were, the more information the choice conveys.

He measured it in units of a single choice between two alternatives, which he called bits, crediting the word to a colleague (Shannon, 1948). He reduced every communication system, from a conversation to a television broadcast, to the same five parts: a source, a transmitter, a channel, a receiver and a destination, with noise entering along the way (Shannon, 1948).

A word, a picture, a piece of music and a set of measurements are, from this point of view, the same kind of thing. They can all be turned into bits, stored, copied and transmitted by the same means. That single abstraction is the reason one device can now be a telephone, a library, a camera and a cinema.

Shannon closes the list fittingly. Panini showed that a language is a system of rules. Shannon showed that anything which can be communicated at all is, at bottom, a sequence of choices.

The pattern across the twenty-five

Set side by side, these lives show a small number of abstract moves used again and again. They are the abilities of Part I, seen at full stretch.

The move	What it is	Who shows it most clearly
Idealisation	Removing what complicates, and stating the law for the pure case	Galileo, Mendel, Turing
Analogy	Carrying a structure from a field where it is understood to one where it is not	Archimedes, Darwin, Maxwell, Shannon
Generalisation	Solving the kind of problem instead of the single problem	Al-Khwarizmi, Noether, Cantor
Formalisation	Laying out a body of knowledge as a system built from stated rules	Panini, Aristotle, Euclid, Gödel
Notation	Inventing signs that carry part of the reasoning	Leibniz, Descartes, Euler
Unification	Seeing that two things thought to be different are one	Newton, Maxwell, Descartes
Naming the unseen	Treating as a real object something inferred but never observed	Brahmagupta, Faraday, Mendel, Curie, Mendeleev
The thought experiment	Reasoning rigorously about a situation that exists only in the mind	Galileo, Newton, Einstein

Five further observations can be drawn from the record.

Almost none of them started from nothing. Darwin needed Malthus and the pigeon breeders. Newton needed Kepler and Galileo. Maxwell needed Faraday, and Einstein needed Maxwell, Gauss and Riemann. Turing needed Gödel. An abstraction, once made, is a tool for whoever comes next, which is the mechanism behind the “limitless” of the title.

> **“Nothing is external.”** Maynard L. Dokken

The decisive step was very often to ignore something. Euler ignored the distances. Shannon ignored the meaning. Galileo ignored the air. Mendel ignored every trait but one. In each case the thing set aside was real and seemed important. Knowing what to leave out is the greater part of the skill.

Many of them thought in pictures. Faraday’s lines, Einstein’s falling man, Newton’s cannonball and Archimedes’ lever are all images. Abstract does not mean without imagery. It means that the image stands for a structure.

They held their abstractions firmly against the evidence of the moment, and gave them up against better evidence. Mendeleev told a chemist that his measurement was wrong, and was right. Kepler abandoned a belief two thousand years old over a quarter of a moon’s width. Judging which case one is in may be the hardest thing of all.

The legends are less interesting than the truth. The bath, the apple, the tower, the dream and the fly are stories of ideas arriving in an instant. The records show long preparation: Darwin’s notebooks, Kepler’s years of arithmetic, Mendel’s thousands of peas. Where there was a sudden insight, it came to a mind that had been filled first, which is what the sleep experiment in Part I found.

Others with a claim

A list of twenty-five leaves out many who could stand in it. Among those whose records were examined for this article and who have a strong claim are Bernhard Riemann, who extended geometry to curved spaces of any dimension; Madhava of Sangamagrama, who found infinite series for π and the trigonometric functions in India about 1400, some two and a half centuries before they were found in Europe; Copernicus; Ada Lovelace, who saw that a calculating engine could work on anything that can be represented by symbols; Ludwig Boltzmann; Paul Dirac, whose equation predicted antimatter before it was observed; John von Neumann; Barbara McClintock; Richard Feynman; and Tim Berners-Lee, who is living, and whose World Wide Web is an abstraction laid over the world's documents.

Next: the same moves, made by a composer, a painter, a film-maker and a designer of games.

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Part III. Where abstract thinking creates

The twenty-five minds of Part II were scientists, mathematicians and inventors. It would be a mistake to conclude that abstract thinking belongs to them. Susanne Langer's distinction (Part I, section 2.8) applies here. The scientist's abstraction generalises: it yields a law that covers every case. The artist's abstraction yields one object, a painting, a fugue or a film, that embodies a structure and lets other people feel it. Both step away from the particular in order to say something that the particular alone cannot.

This part goes through the fields in which abstract thinking has produced extraordinary work, beginning with the arts and entertainment, where its role is least often recognised. It opens with what research has learned about creative achievement in general.

1. What research says about creative achievement

The scientific study of creativity is usually dated from a lecture given in 1950 by the psychologist J. P. Guilford. He distinguished two kinds of thinking. *Convergent* thinking works toward the single right answer, and is what intelligence tests measure. *Divergent* thinking produces many possible answers. Creative work appears to need both, in proportions that depend on the task (Encyclopaedia Britannica, n.d.-d; Sawyer, n.d.).

Four findings since then bear on everything that follows.

Creativity is a judgement made by a community. Mihaly Csikszentmihalyi argued that creativity is found where three things meet: a *person* who produces something new, a *domain* of knowledge, rules and styles within which it counts as new, and a *field* of teachers, critics, editors and peers who decide whether it is admitted. Creativity, in his words, "is not the product of single individuals, but of social systems making judgments about individuals' products" (Csikszentmihalyi, 1999). Hilma af Klint, below, is a striking illustration.

The most successful creators are the most productive ones. Dean Keith Simonton's studies of the careers of eminent creators found that the proportion of a person's works that prove to be important stays roughly constant across a career. Those who produce the most masterpieces are, by and large, those who produce the most work (Simonton, 1997). There are exceptions. As a rule, though, quality comes out of quantity.

Doing it for its own sake matters. Teresa Amabile asked 72 working writers to compose a poem. Some were first asked to dwell on external reasons for writing, such as money and recognition. Their poems

were judged by other poets to be less creative than those of writers who had dwelt on their love of the work, or of writers given no instruction at all (Amabile, 1985).

The greatest scientists are unusually often artists. A study of Nobel laureates in the sciences compared them with scientists in general. The laureates were about three times as likely to practise an art or a craft seriously. They were at least seven times as likely to be visual artists, at least twelve times as likely to write poetry, fiction or plays, and at least twenty-two times as likely to perform as actors, dancers or magicians (Root-Bernstein et al., 2008). The study shows a connection and not a cause. It does suggest that the capacity at work in the laboratory and in the studio is one capacity.

2. Music

Music is abstraction that can be heard. It refers to nothing outside itself, and yet it is as tightly organised as mathematics, to which it has been linked since antiquity.

The foundation is ratio. Two notes an octave apart have frequencies in the ratio two to one. A fifth is three to two. These simple ratios are the reason some combinations of notes sound settled and others do not. They also conceal a difficulty. Twelve fifths placed one on top of another ought to arrive at the same note as seven octaves, and they do not quite: the fifths overshoot by a small amount. The solution adopted in Western music, equal temperament, is itself a feat of abstraction. Every one of the twelve steps in the octave is made exactly the same size, each note higher than the last by the twelfth root of two. No interval is then perfectly pure, and every key becomes usable (Wilson, 2009).

Bach and the fugue. A fugue, in the definition of the *Encyclopaedia Britannica*, is “a compositional procedure characterized by the systematic imitation of a principal theme” (Encyclopaedia Britannica, n.d.-f). One voice states a theme. A second takes it up while the first continues with something that fits against it, and then a third. The composer must think of each line as a melody and of all the lines together as harmony, at the same moment. Johann Sebastian Bach’s fugues are placed by the same source at the very summit of the form. His *Well-Tempered Clavier*, in two books dated 1722 and 1744, contains a prelude and fugue in every one of the twenty-four major and minor keys (Encyclopaedia Britannica, n.d.-f). It is a systematic exploration of the whole space that the new tuning had opened: a demonstration that all of it could be used.

Harmony as a system. In the same year as Bach’s first book, 1722, the French composer Jean-Philippe Rameau published a treatise arguing that all harmony rests on a single principle: every chord has a root, a fundamental note, with the other notes built above it in thirds (Encyclopaedia Britannica, n.d.-o). He did for chords what Euclid did for geometry. A mass of practical knowledge was shown to follow from a few ideas.

Schoenberg’s method. By the early twentieth century some composers felt that the system of keys was exhausted. Arnold Schoenberg replaced it with another, built on a different abstraction. In his Suite for Piano, composed between 1921 and 1923, the music is derived from a fixed ordering of all twelve notes, a row, which is the source of everything else in the piece (Arnold Schönberg Center, n.d.). Whatever one thinks of the sound, the conception is exact: a piece of music generated from a rule.

Jazz. Improvisation looks like the opposite of system, and is in fact one of its highest uses. A jazz musician improvising over a sequence of chords is composing in real time within an abstract framework that all the players hold in their heads. In a small study, six professional jazz pianists played inside a brain scanner, first from memory and then improvising. During improvisation, activity fell in a part of the prefrontal cortex associated with monitoring one’s own performance, and rose in a region at the very front of the brain (Limb & Braun, 2008). The sample was very small and the finding is suggestive only. It fits what musicians report: after years of absorbing the structure, they stop watching themselves and let it work.

3. Painting and the visual arts

Perspective. For most of history, painters had no systematic way of showing depth. Around 1415 the Florentine architect Filippo Brunelleschi devised one, and in 1435 Leon Battista Alberti wrote it down. Lines that are parallel in the world are drawn so as to meet at a single point on the horizon (Encyclopaedia Britannica, n.d.-k). Linear perspective is geometry applied to seeing. It treats the picture as a window and works out, by rule, where each thing must appear on it. It was a discovery in art made by means of mathematics.

Cubism. Five centuries later, artists took perspective apart. Pablo Picasso painted *Les Demoiselles d'Avignon* in 1907, and between then and 1914 he and Georges Braque worked out Cubism together. A Cubist painting breaks an object into geometric planes and shows it from several viewpoints at once (Rewald, 2004). The single window is gone. What is offered instead is closer to what a mind knows about an object than to what an eye sees of it at one instant: an analysis of its structure.

Pure abstraction. Within a few years several painters had taken the further step of removing the object altogether.

Wassily Kandinsky published *Concerning the Spiritual in Art* in Munich at the end of 1911. He argued for painting that expresses inner states directly and does not copy appearances, and he took his terms from music, calling his works impressions, improvisations and compositions (Museum of Modern Art, n.d.).

Kazimir Malevich went furthest in reduction. His *Black Square* is exactly what the title says. It was first shown in December 1915, hung high across a corner of the room, the position an icon would occupy in a Russian home. He later wrote of having tried "desperately to free art from the dead weight of the real world" (Tate, n.d.-a).

Piet Mondrian arrived by stages, abstracting his paintings of trees further and further over a number of years, at a language restricted to straight horizontal and vertical lines, rectangles, the three primary colours, and black, white and grey (Cramer & Grant, 2019; Tate, n.d.-b).

Who painted the first abstract picture is disputed, and may not have an answer (Cramer & Grant, 2019). The dispute now has to include a Swedish painter almost no one had heard of when the history was first written. Hilma af Klint began her large abstract series, *Paintings for the Temple*, in 1906, some years before Kandinsky's first abstract works. She showed them to almost no one and left instructions that they should not be exhibited until twenty years after her death, which came in 1944 (Encyclopaedia Britannica, n.d.-i). Because they were not seen, they influenced no one (Cramer & Grant, 2019). It is Csikszentmihalyi's point exactly. The work existed. Without a field to receive it, it did not enter the history of art for most of a century.

Escher. The Dutch artist M. C. Escher made the link with mathematics explicit. Visits to the Alhambra in Spain, with its interlocking tile patterns, set him on a lifelong study of how a flat surface can be filled completely with repeated shapes. Later he explored impossible spaces, among them a triangle that cannot exist (National Gallery of Art, n.d.). In 1958 the geometer H. S. M. Coxeter sent him a mathematical paper containing a diagram of a curved, non-Euclidean plane. From it Escher made his *Circle Limit* prints, in which figures shrink without end toward the edge of a circle (Dunham, n.d.). It is the geometry that Gauss kept private (Part II), made visible to anyone.

4. Literature

Literature works with the most abstract material of all, since words are already symbols. Its characteristic devices are the abilities described in Part I.

Metaphor is analogy in language: one thing understood through the structure of another. Research on how people understand metaphors suggests that a new one is processed as a comparison, while a familiar one has become a category in its own right, so that we no longer notice the leap (Bowdle & Gentner, 2005). Dead metaphors are abstractions that have been absorbed into the language.

Allegory extends a metaphor across a whole story. It is, in one definition, "a symbolic fictional narrative that conveys a meaning not explicitly set forth in the narrative" (Encyclopaedia Britannica, n.d.-b). Dante's *Divine Comedy* and Bunyan's *Pilgrim's Progress* are journeys that stand for the life of the soul. Orwell's *Animal Farm* is a farmyard that stands for a revolution. In each, a reader holds two structures in mind and maps one onto the other.

The modern novel made abstractions of time and of consciousness. James Joyce's *Ulysses*, published in Paris in 1922, takes place on a single day in Dublin, 16 June 1904. Its characters and episodes correspond to those of Homer's *Odyssey*, so that an ordinary day is laid over an ancient epic, and much of it is told from inside the characters' minds as their thoughts arise (Encyclopaedia Britannica, n.d.-p). Virginia Woolf's *Mrs Dalloway*, published in 1925, is likewise set on one day. She described in her diary a "tunnelling process" by which she brought in the past in pieces, as it was needed (Milthorpe, 2025). Both writers gave up the straightforward sequence of events for a structure that follows the mind.

Borges wrote stories that are thought experiments. "The Library of Babel" imagines a library containing every possible book, every combination of letters, and follows out what would be true of such a place (Kumar, 2015). It is Cantor's subject matter turned into fiction.

A caution is needed about one claim often made for literature. A well-known study in 2013 reported that reading a short passage of literary fiction immediately improves a person's ability to read other people's minds. Most later attempts to repeat that result have failed (FORRT, n.d.; Panero et al., 2016). A more modest finding has held up: across many studies, people who read fiction show a small advantage in understanding others (Dodell-Feder & Tamir, 2018).

5. Film, animation and television

Cinema brought with it a new kind of abstraction, the cut, which the earlier arts did not have.

Montage. The Russian film-maker Lev Kuleshov is reported to have joined the same shot of an actor's expressionless face to shots of a bowl of soup, a child and a coffin. Audiences, the account goes, saw hunger, tenderness and grief in the same face. The meaning was not in either shot. It was made by the join between them. Modern experiments have confirmed that context changes how a face is read, although the effect is less tidy than the legend (Calbi et al., 2017).

Sergei Eisenstein built a theory on it. In *Battleship Potemkin* (1925) he set shots against one another so that a third idea, contained in neither, arose from their collision. The sequence on the Odessa steps is among the most famous in the history of film (Encyclopaedia Britannica, n.d.-c). Montage is relational reasoning carried out with pictures: the viewer supplies the relation.

The storyboard. In the early 1930s the Disney studio began pinning sequences of sketches to large boards so that a whole film could be seen, rearranged and criticised before any of it was animated. The studio credits the idea to one of its story men, Webb Smith (D23: The Official Disney Fan Club, n.d.). The storyboard is a model of a film in the sense that Maxwell's wheels were a model of the field. It is something cheap to change that stands in for something expensive to make. Nearly every film and television programme is now planned in this way.

Animation and the mathematics beneath it. Animated films are abstraction at every level. A character is designed as a set of shapes. Movement is governed by principles that were worked out by hand in the

early studios and later carried over into computer animation (Lasseter, n.d.). Underneath lies mathematics. An educational project created by the Pixar studio sets out some of it: combinatorics to generate crowds of different robots, parabolas to shape blades of grass, weighted averages to model the surface of a character, and trigonometry to build worlds (Motion Picture Association, 2015). A world that has never existed has to be specified completely, since nothing in it can be left to nature.

The creative organisation. The work is also collective. Ed Catmull, a founder of Pixar, has written that creativity in film-making comes from groups and not from single geniuses, and depends on candid communication and on regularly examining what went wrong (Catmull, 2008). That is Csikszentmihalyi's field, built on purpose inside one company.

6. Games

A game is an abstract world with stated rules, and games have long served as a laboratory for the study of thought.

Chess gave psychology its clearest picture of expertise, described in Part I: the master sees the board in large meaningful units and not as separate pieces (Chase & Simon, 1973).

Go supplied one of the most remarkable episodes of recent years. In March 2016 the program AlphaGo played a five-game match in Seoul against Lee Sedol, one of the strongest players in the world, and won four games to one (British Go Association, 2016). In the second game, its thirty-seventh move was one that no experienced player would have chosen. The program's own estimate was that a human would play it one time in ten thousand (Hassabis, 2016). It proved to be the turning point of the game, and a leading professional called the game beautiful (British Go Association, 2016).

What followed is the part that matters for this article. In the fourth game Lee Sedol answered with a move of his own, his seventy-eighth, that was just as unexpected, and won (Hassabis, 2016). And a study of 5.8 million moves played by professionals between 1950 and 2021 found that the quality of human play, which had been level for decades, rose markedly after 2016, with the largest gains in moves that had never been seen before (Shin et al., 2023). The machine did not end human creativity in the game. Human players absorbed new abstractions from it and went on to play better and more inventively than before. Part IV returns to this.

Designing games is the design of systems. The designer Sid Meier described a game as "a series of interesting decisions" (Alexander, 2012). The designer makes no story and no picture, but a space of possibilities for a player to move about in. It is among the most purely abstract of the creative arts, and one of the newest.

7. Engineering, computing and design

Engineering is often taken to be applied science, which undervalues it. The historian Eugene Ferguson argued that an engineer's design takes shape first as a picture in the mind, by a kind of thinking that is visual and not verbal, and that cannot be reduced to calculation. In his account, nothing signals to the designer that the best design has been reached (Ferguson, 1977). The integrated circuit, demonstrated in 1958 and made practical the following year, is an abstraction turned directly into matter: a circuit diagram that has become a physical object (Computer History Museum, n.d.; Nobel Prize Outreach, 2000).

Software is abstraction and nothing else. A program is built in layers, each of which hides the details of the one below so that the next can be written without thinking about them. The pioneers were explicit about this. Edsger Dijkstra, who shaped the discipline of programming, put it in a sentence.

> **“Being abstract is something profoundly different from being vague.”** Edsger W. Dijkstra (Association for Computing Machinery, n.d.-a)

Barbara Liskov was honoured for showing how to organise programs around abstract kinds of data (Association for Computing Machinery, 2009). The internet works because its designers, Vinton Cerf and Robert Kahn, separated the task of moving packets of data from the task of making sure that a message arrives complete, so that each could be solved independently (Association for Computing Machinery, n.d.-b). The software that guided the Apollo spacecraft to the Moon, developed by a team led by Margaret Hamilton, depended on concepts she and her colleagues had to invent for the purpose (MIT News, 2016).

Architecture and design. An architect works almost entirely on abstractions of a building: the plan, the section and the model. A study of architects at work found that their rapid sketches do not record ideas already formed. The sketch produces the idea. The designer draws, sees something unexpected in the drawing, and responds to it (Goldschmidt, 1991). It is thinking carried on outside the head, on paper.

Christopher Alexander proposed in 1977 that good buildings and towns are composed from a vocabulary of recurring solutions, which he called patterns, and he published 253 of them. Architects were sceptical. Programmers adopted the idea, and it reshaped the design of software (Cross, 2022). A structure abstracted in one field became a tool in another, as with every analogy in Part II.

8. Business, economics and law

The institutions of economic life are abstractions so familiar that they are seldom recognised as such. Money is a symbol of value. A company is a person that exists only in law. A contract is a promise given a fixed form. None of these can be seen or touched, and modern life could not go on without them.

Three inventions show the pattern.

Double-entry bookkeeping records every transaction twice, as something received and as something given, so that the books must balance. It was first described in print by the mathematician Luca Pacioli in Venice in 1494 (History of Information, n.d.). It turns the confused life of a business into a system that can be checked for consistency, as a proof can.

Limited liability separates the debts of a company from the property of its owners. When it was made generally available in Britain in the 1850s, its sponsor in Parliament argued that it should be a right open to all and not a privilege granted to a few (UK Parliament, 1856). That legal abstraction made it possible for strangers to pool their money in a common enterprise, and so made possible the large firm.

The pricing of options. In 1973 Fischer Black and Myron Scholes published a formula, extended by Robert Merton, for calculating the value of a contract that gives the right to buy something at a fixed price in the future. Scholes and Merton received the 1997 prize in economics for it. Within a few years thousands of traders were using the formula every day, and a vast market grew up on it (Nobel Prize Outreach, 1997). An equation had created a market.

9. Medicine and public health

Some of the abstractions that have saved the most lives were not treatments. They were ways of thinking.

The germ theory proposed that many diseases are caused by living things too small to see. Pasteur’s work on fermentation, Lister’s antiseptic surgery in the 1860s and Koch’s identification of the organisms of tuberculosis and cholera in the 1880s established it (Encyclopaedia Britannica, n.d.-g). It is a case of

naming the unseen, like Mendel's factors and Faraday's lines, and it was resisted in the same way: surgeons were still operating in bloodstained coats in the 1890s (Encyclopaedia Britannica, n.d.-g).

The map as an argument. When cholera broke out in London at the end of August 1854, the physician John Snow marked the deaths on a street map. They clustered around one water pump, in Broad Street. He put the evidence to the local authorities on 7 September, and the pump was closed the next day (Ashworth, 2017). Snow did not know what in the water caused the disease. By abstracting one fact about each death, its location, and setting those facts in a new arrangement, he revealed a cause that no examination of any single patient could have shown.

The randomised trial. How can it be known whether a treatment works, when patients differ and doctors hope? The answer is to let chance decide who receives it. In the trial of the drug streptomycin for tuberculosis, published in Britain in 1948, patients were assigned to treatment or to comparison by random numbers kept in sealed envelopes, so that no one's judgement could tilt the result (Chalmers, 2010). The idea is a pure abstraction. It deliberately ignores everything known about each patient in order to learn something true about the treatment.

10. Sport, strategy and everyday life

Even in sport, the most physical of activities, abstraction has altered what is done. From about 1980 the writer Bill James and others began analysing baseball statistically, asking which measurable events actually lead to winning. Professional teams adopted the approach, and comparable analysis has since spread through nearly every spectator sport (Neyer, n.d.).

In ordinary life the same abilities are at work in smaller matters: seeing that a present difficulty has the same shape as an earlier one, stating the principle behind a decision, weighing what has not happened but could. Two findings from research on education say how these abilities are best developed.

The first is called concreteness fading. Reviewing the research on the teaching of mathematics and science, one group of authors recommends "beginning with concrete materials and then explicitly and gradually fading to the more abstract" (Fyfe et al., 2014). Neither the concrete alone nor the abstract alone works as well. This is the order Bruner described half a century earlier (Part I, section 3.3), and the caution Vygotsky gave: a symbol that has not been connected to experience is empty.

The second is comparison. People learning to negotiate were given two worked cases. Those asked to compare the two and say what they had in common were far more likely to use the underlying principle later, in a real negotiation, than those who studied the same two cases one after the other (Gentner et al., 2003). It is the infant experiment of Part I, repeated with adults. An abstraction is drawn out by setting examples side by side.

11. One capacity, many fields

The fields in this part look unlike one another, and the same few moves appear in all of them.

The move	In science (Part II)	In the arts and other fields (Part III)
Idealisation: removing in order to reveal	Galileo's fall without air	Mondrian's lines and primary colours; Malevich's square
Analogy: one structure laid over another	Darwin's breeder and nature	Joyce's Dublin laid over Homer; allegory

The move	In science (Part II)	In the arts and other fields (Part III)
System from a few rules	Euclid’s axioms	Rameau’s harmony; the fugue; the twelve-note row
Notation and models as tools for thought	Leibniz’s symbols	The musical score; the storyboard; the architect’s sketch
Changing the viewpoint	Einstein’s observers	Perspective, and then Cubism
Meaning from relation and not from the parts	Shannon’s information	Kuleshov’s and Eisenstein’s montage
Rearranging the data to reveal a cause	Mendeleev’s table	Snow’s map

The laureates who paint and write, the mathematician’s diagram that became Escher’s prints, the architect’s patterns that reshaped software: these crossings between fields are not coincidences. Abstract thinking is not a scientific skill that artists sometimes borrow, nor an artistic gift that scientists sometimes share. It is a single human capacity, and each field is one of the places where it has been put to work.

That is why creativity and the arts are not an ornament to be added once the serious work is done. They are the same capacity, exercised where there is no formula to fall back on, and a society that neglects them is training half of its mind. The future, here and beyond this planet, will be built by people who can imagine what does not yet exist.

> **“We will never get there if politics and greed are the only motivators.”** Maynard L. Dokken

Next: the newest tool of the abstract mind, and what separates using it well from using it badly.

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Part IV. A new tool for the abstract mind

Every tool described in this article, from the written word to the equation, extended what a mind could do. This part is about the newest of them. It sets out what the evidence shows about thinking with the help of artificial intelligence: when it helps, and when it harms.

The position taken here can be put in a sentence. **AI assistance does not weaken a thinker who uses it rightly. It speeds the development of ideas and their communication. It weakens the thinker only when it is allowed to do the thinking.** The evidence supports both halves of that sentence, and both are given here.

1. The oldest fear about the newest tool

The fear that a tool for thinking will ruin the thinker is about 2,400 years old. In Plato’s *Phaedrus*, Socrates tells of the Egyptian god who invented writing and offered it to the king as an aid to wisdom. The king refused the claim. “This discovery of yours,” he says, “will create forgetfulness in the learners’ souls, because they will not use their memories.” Those who rely on it “will appear to be omniscient and will generally know nothing” (Plato, n.d.).

The charge is the one now brought against artificial intelligence, almost word for word: that people will seem to know what they do not know, and will lose the power they have handed over. It deserves to be

taken seriously, and section 3 shows that it is partly right. It should also be noticed that we know of the king's warning only because Plato wrote it down.

Every part of this article has shown the other side of the matter. The human workshop is small (Part I, section 9), and thought has advanced by building tools that hold what the mind cannot. Whitehead said it of notation in 1911: "By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems" (Whitehead, 1911). Leibniz's symbols, Descartes's coordinates and Euler's notation were tools of that kind. None of them made mathematicians worse at mathematics.

The nearest recent parallel is the electronic calculator, which many feared would destroy children's ability to do arithmetic. The matter was studied for decades. Two pooled analyses of the research, one of 79 studies and one of more than 50, agree: pupils who used calculators did not, in general, lose their skills, and they became better at solving problems and more confident (Boyle & Farreras, 2015; Ellington, 2000). The benefit depended on how the calculator was used. Teachers of mathematics now hold that it helps when it serves a purpose in the lesson and does not replace mental and written methods (National Council of Teachers of Mathematics, 2015).

A calculator, though, is always right, and it does not do the reasoning. An AI assistant can be wrong, and it can do the reasoning, or appear to. So the comparison encourages us without settling anything. The question has to be answered from studies of the new tool itself.

2. The idea is older than the machine

That a computer might be a partner in thought was proposed before any computer could do it. In 1960 the psychologist J. C. R. Licklider published an essay called "Man-Computer Symbiosis". He had kept a record of how he spent his own working time and found that about 85 per cent of the hours he called thinking went on getting into a position to think: finding information, plotting data, preparing the ground. He imagined a partnership in which the machine would do that preparing, and he was exact about the division of labour. "Men will set the goals, formulate the hypotheses, determine the criteria, and perform the evaluations" (Licklider, 1960).

Two years later the engineer Douglas Engelbart set out a programme for what he called augmenting human intellect, by which he meant "increasing the capability of a man to approach a complex problem situation", to understand it and to find solutions (Engelbart, 1962). To show that tools shape thought, he tied a pencil to a brick and had people write with it. The clumsier the tool, the cruder the thinking.

> **"Men will set the goals, formulate the hypotheses, determine the criteria, and perform the evaluations."** J. C. R. Licklider, 1960

The same case was being made in business forty years later. In 2000, as chief executive of Milinx, I spoke on the programme *World Business Review* about what I called intelligent applications (Milinx, 2000): software that would not wait to be operated, and would do part of the work of finding, assessing and presenting what a person needed to know. The applications of that time could not yet do it. The argument was the one Licklider and Engelbart had made, and it is the one this part makes now.

Sixty years on, Licklider's division of labour is the best short account available of how to use an AI assistant well. The person sets the goals, forms the hypotheses, fixes the standards and judges the results. That is what the research finds, as the next section shows.

3. What the evidence shows

Enough careful studies now exist to say something definite. Most are single studies that have not yet been repeated, and the tools change quickly, so every figure below should be read as a first measurement and not as a settled fact.

3.1 The gains

Writing. In an experiment with 453 professionals, among them marketers, analysts, consultants and managers, half were given an AI assistant for a writing task typical of their work. They finished about 40 per cent faster, and their work was rated 18 per cent better by experienced judges who did not know which pieces had been assisted. The weakest writers gained the most (Noy & Zhang, 2023).

Professional analysis. In a trial with 758 consultants at a large international firm, those given an AI model completed more tasks, completed them about 25 per cent faster, and produced work rated more than 40 per cent higher in quality, on tasks that lay within the model's abilities (Dell'Acqua et al., 2023).

Service work. Among more than 5,000 customer-support agents, an AI assistant raised the number of problems resolved per hour by 14 per cent on average, and by 34 per cent for new and less skilled workers. The authors suggest that the tool spread the practices of the best workers to everyone else (Brynjolfsson et al., 2025).

Ideas. Writers offered ideas by an AI produced stories that were judged about 8 per cent more novel and 9 per cent more useful. Here too the least creative writers gained most and were brought level with the most creative (Doshi & Hauser, 2024).

Learning. In a university physics course, students taught by an AI tutor learned more than twice as much, in less time, as students in a well-run class. The tutor had been built to lead them through a problem one step at a time and not to hand over the answer (Kestin et al., 2024).

Taken together these are large effects. They support the first half of the sentence above. Used on the right work, the tool makes people faster and better, and it helps most those who start furthest behind.

3.2 The harms

The same studies, and others, show what goes wrong.

Trusting it beyond its ability. The consultants were also given a task chosen to lie just outside what the AI could do well. On that task, those using the AI were 19 percentage points *less* likely to reach the right answer than those working alone. They accepted a persuasive wrong answer. The authors describe the abilities of these systems as a "jagged technological frontier": remarkably good at some tasks and poor at others that look no harder, with nothing to mark where the edge runs (Dell'Acqua et al., 2023).

Handing over the work. In the writing experiment, a third of those given the assistant submitted its first output without changing a word, and most of the others edited only lightly (Noy & Zhang, 2023). They were faster, and they had contributed almost nothing.

Leaning on it while learning. Nearly a thousand secondary-school pupils practised mathematics with an AI. Those given an ordinary chatbot did much better during practice, and then, when it was taken away, scored 17 per cent *worse* in the examination than pupils who had never had it. They had used it as a crutch. A second group was given a version designed to give hints and not answers, and largely escaped the harm (Bastani et al., 2025). This is the king's warning from the *Phaedrus*, measured.

Feeling helped when one is not. Sixteen experienced programmers, working on software they knew well, took 19 per cent longer on tasks when allowed to use AI tools. They had expected to be faster, and

afterwards they still believed they had been (Becker et al., 2025). A person's own sense of whether the tool is helping is not reliable.

Trusting the tool more than oneself. A survey of 319 people who use AI in their work found that the more confidence they had in the AI, the less critical thinking they reported doing. The more confidence they had in their own ability, the more they did (Lee et al., 2025).

Everyone having the same ideas. The stories written with AI ideas were individually better and collectively more alike (Doshi & Hauser, 2024). A study of more than forty million scientific papers found the same pattern on a larger scale: scientists who used AI methods published and were cited far more, while the range of topics studied grew narrower (Hao et al., 2025). Each person gains, and the common stock of ideas may shrink.

Weak judgement made weaker. When small-business owners were given an AI adviser, those who were already doing well gained about 15 per cent. Those who were struggling did about 8 per cent worse. The difference did not lie in the advice they received. It lay in how they chose among the suggestions and acted on them (Otis et al., 2023).

Two further cautions belong here. A widely cited analysis claiming that AI chatbots greatly improve learning was withdrawn in 2026 because of faults in its figures, and a much-publicised paper on AI and scientific discovery was disowned by the university it came from (Plagiarism Today, 2026; TechCrunch, 2025). The enthusiasm in this field sometimes runs ahead of the evidence, and this article has tried not to add to it.

3.3 What "used rightly" means

Set the gains beside the harms and the dividing line is clear. It is not drawn between people who use the tool and people who do not. It is drawn between two ways of using it.

What helped	What harmed
Using it on work it does well, and knowing where that stops	Trusting it on work beyond its ability
Keeping the choosing and the judging in one's own hands	Accepting its output unedited
Checking what it produces	Relying on the feeling of being helped
Tools and habits that guide thinking instead of replacing it	Using it as a crutch while learning
Thinking first, then turning to the tool	Trusting the tool more than oneself
Bringing one's own ideas to it	Drawing one's ideas from it, as everyone else does

Every item on the left keeps the person in the place Licklider gave them: setting the goals, forming the hypotheses, fixing the standards, judging the results. Every item on the right gives one of those up.

In the language of Part I, the tool is safe wherever it extends the workshop and dangerous wherever it is asked to replace the worker. A calculation, a search through the literature, a first arrangement of material, a check for errors, a draft to be torn apart: these are the "getting into a position to think" on which Licklider found he spent most of his time. The abstraction itself is another matter. Seeing which relation matters, deciding what to ignore, noticing that one thing has the structure of another, judging whether the result is true: these have to remain with the person, because they are the thinking.

The game of Go, described in Part III, shows what is possible. A machine made a move no human would have made. The best players studied it, and the measured quality and originality of human play rose. The tool gave them new abstractions, and they built on them, as Newton built on Kepler.

There is a link to an idea from Part I as well. Vygotsky held that a learner can do more alongside a more capable partner than alone, and that what is first done with help is later done without it. A tool that guides, questions and checks can stand in that partner's place. The school experiment shows the condition attached. The help has to be of the kind that builds the learner's own ability, and not of the kind that stands in for it.

4. The same capacity, with a longer reach

Part I showed that the mind has a small working space and no fixed limit, because whatever it understands it can fold into a single unit and build on. Every tool in this article lengthened that reach: writing, notation, the diagram, the printed book and the computer. Each was feared when it was new. Each, used well, let people think what could not have been thought before.

An AI assistant is the newest form of these systems, and the first that can answer back with context. That makes it a faster and more useful means of assessing work, and of reaching knowledge, than searching out and retaining outside sources for every review. It can also be misused, through carelessness, as the studies above show, and by those with bad intentions. That is the dilemma, and the answer to it has not changed since Licklider: the person sets the goals and judges the results.

Next: the same thinking at work outside the laboratory and the studio, in a series of ventures over four decades.

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Part V. The thinking at work: a summary

The first four parts were about other people. This part is a short account of the same thinking at work in a series of ventures that I founded or led between 1987 and today. It is a summary only. Each venture will have an article of its own in the series that follows this one, and each is described in more detail under Major Milestones on this site.

When I built these companies I had never heard the terms used in Part I. I only knew how I worked. Faced with a problem, I did not start with the problem. I stepped back and asked what kind of problem it was, and whether I had seen its shape somewhere else. The table shows what that produced each time: on the left, the particular problem as everyone saw it, and on the right, the more general thing I took it to be.

> **"I did not start with the problem. I asked what kind of problem it was."** Maynard L. Dokken

Venture	Years	The particular problem	The abstraction
SuccessInc	1987–1994	Every user needed the software on their own machine	The application is a service. Run it once, centrally, and let people reach it over a network
Assured Card	from 1994	One missed payment set off interest and fees that fell hardest on ordinary borrowers	A missed payment is a gap in timing. Cover the gap, and the spiral does not start

Venture	Years	The particular problem	The abstraction
Assured Credit	from 1996	Enrolment on paper was costly and insecure	A payment is a secure message. Build the terminal and the gateway that carry it, with the customer’s data kept on the customer’s side
Milinx Business Group	1997–2002	A business needed its own servers, software and staff to run them	The whole office is a service: desktop, messaging and business applications delivered over the web, and to mobile devices
MarkCrest	from 2002	Assets that cannot be sold or moved earn their owners nothing	Value can be separated from the asset that holds it, and given liquidity under rules that protect the owner
SharedMercantile	from about 2006	An investor holds a paper claim, and others control the asset	Own the thing itself, and earn from it directly
RIF Industries	from about 2007	Value-added supply chains are controlled by a few large concerns	A supply chain is a sequence of steps that anyone can take part in: buy a base product, process it, deliver something worth more
Swomi	since 2015	The value of what people write and share is kept by the platforms	A person’s words, and the audience that trusts them, are assets that belong to that person

One line of thought

Set out in order, these are not eight separate ideas. They are one idea followed for forty years, and each step was built on the one before, in the way described at the end of Part I.

From software to service (1987). SuccessInc built a framework called AppNet that ran an application in one place and delivered it to users over modem lines, through a shared gateway and a shared database. Its first application, in 1988, was online recruitment. The model was later given names: the application service provider, then software as a service, then the cloud. The abstraction was to separate what a program does from where it runs.

From service to trust (1994–1996). A service that handles money has to be trusted. Assured Card served about ten thousand customers between 1994 and 1999. Because the open web was not yet safe for financial data, it was reached through a closed terminal: a specially packaged browser connecting over an encrypted channel. Assured Credit was founded to build that technology, and it produced a browser-based payment terminal and a hosted payment gateway, on a design in which the customer’s information stays under the customer’s control. The design was set out in a patent filed in 2000. The abstraction was to treat a payment as a message that must arrive intact and unread.

From one service to all of them (1997). Milinx combined the two. It was a public company that built a data centre and a platform for delivering a whole suite of business applications over the web, including an early hosted desktop and a standard for applications on mobile devices. It did not survive the downturn of 2001 and 2002. The idea did, and it is how most business software is now delivered.

From information to value (2002–2009). If a service and a payment can be separated from the machine and the paper that carry them, so can value. MarkCrest's central work was a framework for taking assets that are illiquid or restricted, and giving their owners liquidity and a return under governed terms. SharedMercantile took the next step and removed the paper altogether: the individual owns the physical asset. RIF Industries turned that model into a working platform for supply chains, until an accident in 2009 interrupted it.

From value to the person who creates it (2012 onward). Swomi applies the same reasoning to the most common asset of all. What a person writes, recommends and shares has value, and so does the trust of the people who listen. The principle is "Your Words Have Worth". It was put on the public record in patent filings in 2012, and it is being built today as a set of products that return that value to the person who created it.

> **"Separate the thing from what carries it. That is the move, made five times."** Maynard L. Dokken

What the series will cover

Each of these deserves its own account: the problem as it looked at the time, the abstraction that answered it, what was built, what failed and why, and what it led to. The articles that follow this one will take them in order, and each will be paired with a piece on the thinking behind it.

Next in this series: SuccessInc, and the idea that software could be a service, in 1987.

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Closing: the limitless mind

A child of three looks at a small room and understands that it stands for a large one. Everything in this article has been a longer version of that moment.

Part I showed what the ability is made of: seeing relations, generalising, using symbols, forming concepts, imagining what has not happened, and reflecting on one's own thought. It showed that the ability is present before speech, that it is drawn out by teaching and by the habits of a whole culture, that it runs on identifiable systems in the brain, and that it can be lost to injury, illness, exhaustion and stress. It showed that the working space of the mind is very small, about four things at a time, and that abstraction is the means by which so small a space takes in so much.

Part II showed what the ability has done. A grammarian saw that a language is a set of rules. A geometer derived a subject from five assumptions. A physicist imagined a fall with no air, and another imagined riding beside a beam of light. A friar counted peas and inferred the unit of heredity. An engineer set meaning aside and found that he could measure information. In each case someone stepped back from the particular thing and saw a structure, and the world was changed by what they saw.

Part III showed that the ability is not the property of science. The fugue, the vanishing point, the cut between two shots, the allegory, the balance sheet and the map of an epidemic are all abstractions, made by the same moves.

Part IV showed that a new tool has arrived which, used rightly, lengthens the reach of this ability, and which, used wrongly, takes its place.

Part V showed the same moves at work in building things: a service delivered over a network, a payment made safely, value given back to the person who created it.

The word in the title can now be given its exact meaning. The mind is not without limits. Its memory, its attention and its hours are short. What has no limit is the height to which abstractions can be built on one another. Brahmagupta's zero is in al-Khwarizmi's algebra. That algebra is in Descartes's geometry, which is in Newton's laws, which are in Maxwell's equations, which are in Einstein's. Every one of those steps was taken by a mind holding about four things at once.

> **"We are in the microverse, and just discovering what the macroverse may be."** Maynard L. Dokken

No one knows what the next step is. The evidence gathered here suggests where it will come from: from someone who knows a field deeply enough to see past its surface, who notices that a structure from somewhere else fits, who has the nerve to ignore what everyone else regards as essential, and who checks the result against the world. That capacity is not reserved for the twenty-five. It is the one the three-year-old used on the model room, and it is in everyone who reads this.

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